Grad, Div, Curl, and all that¼

Reading: Mathews & Walker, *Mathematical Methods of Physics*, ch. 15.

S. Weinberg, *Gravitation and Cosmology*, ch. 3 & 4.

Ohanian and Ruffini, *Gravitation and Spacetime*, ch. 6 & 7.

McConnell, *Applications of Tensor Analysis*, Ch. 12.

The generally covariant differential operators

The gradient operator is obvious, and we have already derived it. If $\varphi\,(x)$ is a scalar field, then

$$\varphi_{,\,\mu} \overset{df}{=} \frac{\partial\varphi}{\partial x^{\mu}} \tag{18.1}$$

in fact transforms as a covariant vector under general coordinate transformations.

Next, consider the covariant curl defined by

$$\begin{aligned} \mathrm{curl}_{\nu}\, A_{\mu} &\overset{df}{=} A_{\mu\,;\,\nu} - A_{\nu\,;\,\mu} = A_{\mu\,,\,\nu} - A_{\nu\,,\,\mu} - \left[\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} - \begin{Bmatrix}\lambda\\ \nu\mu\end{Bmatrix}\right] A_{\lambda} \\ &\equiv A_{\mu\,,\,\nu} - A_{\nu\,,\,\mu}\ . \end{aligned} \tag{18.2}$$

Finally, the divergence should generalize the flat-space result $\mathrm{div}\ V \overset{df}{=} V^{\mu}{}_{,\,\mu}$:

$$\mathrm{div}\ V \overset{df}{=} V^{\mu}{}_{;\,\mu} \equiv V^{\mu}{}_{,\,\mu} + \begin{Bmatrix}\mu\\ \mu\sigma\end{Bmatrix} V^{\sigma}\,. \tag{18.2}$$

However,

$$\begin{Bmatrix}\mu\\ \mu\sigma\end{Bmatrix} = \tfrac{1}{2}\, g^{\mu\lambda}\left[g_{\mu\lambda\,,\,\sigma} + g_{\lambda\sigma\,,\,\mu} - g_{\mu\sigma\,,\,\lambda}\right] \equiv \tfrac{1}{2}\, g^{\mu\lambda}\; g_{\mu\lambda\,,\,\sigma} \tag{18.3}$$

where we have used the antisymmetry in $\mu\lambda$ of the terms $g_{\lambda\sigma\,,\,\mu} - g_{\mu\sigma\,,\,\lambda}$ to drop them after contraction with $g^{\mu\lambda}$.

Now consider $g_{\mu\nu}$ as a matrix G, with $dG = g_{\mu\nu\,,\,\sigma}\, dx^{\sigma}$; then

$$\begin{Bmatrix}\mu\\ \mu\sigma\end{Bmatrix} dx^{\sigma} = \tfrac{1}{2}\,\mathrm{Trace}\left[G^{-1}\, dG\right]. \tag{18.4}$$

Let us call $\det(G) = -g$ (this is a standard notation, with the – sign introduced to make g positive); then

$$-g = e^{\mathrm{Trace}\,[\,\log(G)\,]} \tag{18.5}$$

(Eq. 18.5 is far from obvious, useful, and *worth remembering*. The proof is given below. For now, just believe it!)

Thus

$$\log(-g) = \mathrm{Tr}\left[\log(G)\right]$$

and

$$\mathrm{Tr}\left[\log(G + dG)\right] = \mathrm{Tr}\,[\log(G)] + \mathrm{Tr}\left[\log(1 + G^{-1}dG)\right],$$

hence

$$\log\left(-g - dg\right) = \log\left(-g\right) + \mathrm{Tr}\left[G^{-1}dG\right]$$

so that

$$\begin{Bmatrix}\mu \\ \mu\sigma\end{Bmatrix} dx^\sigma = \tfrac{1}{2}\, d\left[\log(-g)\right] \equiv d\left[\log(\sqrt{-g})\right]$$

or (we can now drop the – sign from $\sqrt{-g}$)

$$\begin{Bmatrix}\mu \\ \mu\sigma\end{Bmatrix} = \partial_\sigma \log(\sqrt{g}\,) = \frac{1}{\sqrt{g}}\partial_\sigma \sqrt{g}\ . \tag{18.6}$$

"So what?" you may say, "I've got my own troubles." Here's what:

Remember Eq. 18.2? Now we may write

$$V^\mu{}_{;\mu} = V^\mu{}_{,\mu} + \frac{1}{\sqrt{g}} V^\mu \partial_\mu \sqrt{g} \equiv \frac{1}{\sqrt{g}}\partial_\mu\left(V^\mu\sqrt{g}\right). \tag{18.7}$$

That is, the expression for the covariant divergence is charmingly simple.

The divergence theorem

Under coordinate transformations, the volume element changes like

$$d^n x \rightarrow d^n\bar{x}\, \det\left(\frac{\partial x}{\partial \bar{x}}\right) \tag{18.8}$$

where $\det\left(\frac{\partial x}{\partial \bar{x}}\right)$ isthe Jacobian of the transformation.

But

$$\bar{g}_{\mu\nu} = g_{\kappa\lambda}\frac{\partial x^\kappa}{\partial \bar{x}^\mu}\frac{\partial x^\lambda}{\partial \bar{x}^\nu} \tag{18.9}$$

so, using a well-known property of determinants of matrix-products,

$$\det(AB) = \det(A)\det(B)\,, \tag{18.10}$$

we find

$$\bar{g} = g\left[\det\left(\frac{\partial x}{\partial \bar{x}}\right)\right]^2 \tag{18.11}$$

i.e.

$$\sqrt{\bar{g}}\, d^n x = \sqrt{g}\, d^n\bar{x}\,. \tag{18.12}$$

In other words, $\sqrt{g}\, d^n x$ is the invariant volume element in a general space of n dimensions.

Now we can transform the volume integral of a divergence into an integral of a vector over a (normal) hypersurface.

$$\int \text{div}\, V \sqrt{g}\, d^n x \;=\; \int V^\mu{}_{;\,\mu} \sqrt{g}\, d^n x \;=\; \int d^n x\, \partial_\mu \left(\sqrt{g}\, V^\mu\right) \equiv \int dS_\mu\, V^\mu \sqrt{g}\,. \tag{18.13}$$

Proof of the theorem about determinants:

We want to prove that for some matrix G,

$$\log\,[\det(G)] \;=\; \text{Tr}\,[\log(G)]\;.$$

Now this is obvious if the matrix is diagonalizable, with eigenvalues g_k since then

$$\log\,[\det(G)] \;=\; \log\left[\prod_{k=1}^{N} g_k\right] \equiv \sum_{k=1}^{N} \log(g_k)$$

and

$$\text{Tr}\,[\log(G)] \;=\; \sum_{k=1}^{N} \log(g_k)\;.$$

We are concerned to prove the theorem more generally. First, it had better be true that the matrix

$$A \;\overset{df}{=}\; \log(G)$$

exists (that is, it can be defined, the matrix has no zero eigenvalues, *etc. etc.*). Assuming this is the case, let

$$G(\lambda) \;\overset{df}{=}\; e^{\lambda A}, \qquad\qquad G(1) \;=\; e^A \;=\; G\,.$$

Now let us define

$$\begin{aligned} d\log\,[\det(G(\lambda))] \;&\overset{df}{=}\; \log\,[\det(G(\lambda + d\lambda))] \;-\; \log\,[\det(G(\lambda))] \\ &=\; \log\,[\det(G + dG)] \;-\; \log\,[\det(G)] \\ &=\; \log\,[\det\!\left(G(1 + G^{-1}dG)\right)] \;-\; \log\,[\det(G)] \\ &=\; \log\,[\det(1 + G^{-1}dG)]\;. \end{aligned}$$

Now let us compute the last term:

Proof of the theorem about determinants:

$$\det(1+G^{-1}dG) = \begin{vmatrix} 1+(G^{-1}dG)_{11} & (G^{-1}dG)_{12} & (G^{-1}dG)_{13} & \cdots \\ (G^{-1}dG)_{21} & 1+(G^{-1}dG)_{22} & (G^{-1}dG)_{23} & \cdots \\ (G^{-1}dG)_{31} & (G^{-1}dG)_{32} & 1+(G^{-1}dG)_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

$$= \prod_k \left[1+(G^{-1}dG)_{kk}\right] + O\left((G^{-1}dG)^2\right)$$

$$\approx 1 + \sum_k \left(G^{-1}dG\right)_{kk} \equiv 1 + \mathrm{Tr}\left(G^{-1}dG\right).$$

To this same order, then,

$$d\log\left[\det(G(\lambda))\right] = \log\left[1 + \mathrm{Tr}\left(G^{-1}dG\right)\right] = \mathrm{Tr}\left(G^{-1}dG\right)$$

However, since $G(\lambda) = e^{\lambda A}$, clearly

$$dG(\lambda) = Ae^{\lambda A}\,d\lambda$$

and thus

$$d\log\left[\det(G(\lambda))\right] = \mathrm{Tr}\left(G^{-1}dG\right) = \mathrm{Tr}\left(e^{-\lambda A}\,A\,e^{\lambda A}\right)d\lambda \equiv \mathrm{Tr}(A)\,d\lambda\ ,$$

giving, by direct integration,

$$\log\left[\det(G(\lambda))\right] = \lambda\,\mathrm{Tr}(A) + \text{constant}\ .$$

Since both sides must vanish when $\lambda = 0$, the constant is zero, giving at last (with $\lambda = 1$)

$$\det(G) = \exp\left[\mathrm{Tr}(A)\right] = \exp\left[\mathrm{Tr}(\log(G))\right]\ .$$