The curvature tensor

**Covariant derivative of contravariant vector**

The covariant derivative of a (contravariant) vector is

\[ V^\mu;_\nu = \partial^\nu V^\mu + \left\{ \frac{\mu}{\nu\sigma} \right\} V^\sigma. \tag{19.1} \]

We used this in Eq. 18.2 without explaining it. Where does it come from? We know that the derivative of a scalar is a covariant vector,

\[ \varphi,_{\mu} = \partial_{\mu} \varphi. \]

Now, suppose the scalar is the contraction of 2 vectors:

\[ \varphi = A^\mu B^\mu \tag{19.2} \]

then by definition (and the product rule)

\[ \partial_{\nu} \varphi = A^\mu,_{\nu} B^\mu + A^\mu B^\mu,_{\nu} \equiv A^\mu,;_{\nu} B^\mu + A^\mu B^\mu;_{\nu} \]

\[ = \begin{bmatrix} \left[ A^\mu,_{\nu} + \left\{ \frac{\sigma}{\nu\mu} \right\} A^\sigma \right] \end{bmatrix} B^\mu + A^\mu B^\mu;_{\nu} \tag{19.3} \]

From Eq. 19.3 we have

\[ A^\mu,_{\nu} B^\mu + A^\mu B^\mu,_{\nu} = A^\mu,_{\nu} B^\mu + \left\{ \frac{\sigma}{\nu\mu} \right\} A^\sigma B^\mu + A^\mu B^\mu;_{\nu} \]

or

\[ A^\mu B^\mu;_{\nu} = A^\mu,_{\nu} B^\mu + \left\{ \frac{\sigma}{\nu\mu} \right\} A^\sigma B^\mu = A^\mu \left[ B^\mu;_{\nu} + \left\{ \frac{\mu}{\nu\sigma} \right\} \right] B^\sigma \tag{19.4} \]

and since \( A^\mu \) is arbitrary, we may say

\[ B^\mu;_{\nu} = B^\mu;_{\nu} + \left\{ \frac{\mu}{\nu\sigma} \right\} B^\sigma \tag{19.5} \]

**Covariant derivative of tensor**

By similar manipulations, we can identify the covariant derivative of a contravariant second-rank tensor—we write

\[ \varphi = A^\mu B^\nu T^{\mu\nu} \tag{19.6} \]

and use the product rule again to write

\[ \partial_{\kappa} \varphi = A^\mu,_{\kappa} B^\nu T^{\mu\nu} + A^\mu B^\nu,_{\kappa} T^{\mu\nu} + A^\mu A^\nu B^\mu T^{\mu\nu},_{\kappa} \]

\[ \equiv A^\mu,_{\kappa} B^\nu T^{\mu\nu} + A^\mu B^\nu,_{\kappa} T^{\mu\nu} + A^\mu A^\nu B^\mu T^{\mu\nu},_{\kappa} \tag{19.7} \]

to find
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Geodesic coordinates

\[ T^{\mu \nu ; \kappa} = T^{\mu \nu , \kappa} + \left\{ \begin{array}{l} \mu \\ \kappa \sigma \end{array} \right\} T^{\sigma \nu} + \left\{ \begin{array}{l} \nu \\ \kappa \sigma \end{array} \right\} T^{\mu \sigma} \]  

(19.8)

and so forth.

Geodesic coordinates

Suppose we locally change coordinates to a system \( x'^\mu = b'^\mu_\sigma x^\sigma \), with (linear) transformation coefficients \( b'^\mu_\sigma \) chosen such that the new metric at that point is

\[ g'^{\mu \nu} = g^{\alpha \beta} b'^\mu_\alpha b'^\nu_\beta = \eta^{\mu \nu}. \]  

(19.9)

Equations 19.9 constitute 10 inhomogeneous equations for 16 unknowns \( b'^\mu_\sigma \), whose determinant,

\[ \det[ g'^{\alpha \beta} ] = \frac{1}{-g}, \]

is non-zero. Therefore they can always be solved, leaving 6 free parameters. These are in fact the 6 parameters of the Lorentz transformation (3 boost, 3 rotation) which, as we already know, leave the Minkowski metric unchanged.

Moreover, we can specify the coordinates further so that in the new system, all first derivatives of the new metric, \( g'^{\mu \nu , \kappa} \) vanish at the point \( a^\sigma \). The coordinates that do this are

\[ x'^\mu = b'^\mu_\sigma x^\sigma + \frac{1}{2} \Gamma'^\mu_\sigma^\alpha (x^\sigma - a^\sigma)(x^\kappa - a^\kappa) + \]

\[ + \frac{1}{3!} \Lambda'^\mu_\sigma^\alpha \beta \lambda (x^\sigma - a^\sigma)(x^\kappa - a^\kappa)(x^\lambda - a^\lambda) + ... \]

where the coefficients \( \Gamma'^\mu_\sigma^\alpha \) and \( \Lambda'^\mu_\sigma^\alpha \beta \lambda \) are manifestly symmetric in their lower indices, hence represent 20 and 80 independent parameters, respectively.

Problem:

An object with 3 indices that run from 0 to 3 obviously has 64 components. Show that if the object is fully symmetric in the 3 indices, then there are but 20 independent components.

Hence show that \( \Lambda'^\mu_\sigma^\alpha \beta \lambda \) has 80 independent components.

Problem:

Find the relation between the coefficients \( \Gamma'^\mu_\sigma^\alpha \) and the Christoffel symbols \( \left\{ \begin{array}{l} \mu \\ \sigma \kappa \end{array} \right\} (a) \) defined at the point \( a^\sigma \) in terms of the (derivatives of the) old metric \( g'^{\mu \nu} \).

Since there are 20 first derivatives of the metric tensor, we can obviously choose the coefficients \( \Gamma'^\mu_\sigma^\alpha \) to set the derivatives of the new metric tensor equal to zero at one point. Since then the
Christoffel symbol vanishes, the motion of a test body in these new coordinates is unaccelerated, i.e. freely falling.

What about the second derivatives of the metric tensor? There are 100 distinct components, $g'_{\mu\nu}, \kappa\sigma$, (two distinct pairs of symmetric indices, giving $10 \times 10 = 100$), but only 80 independent parameters $\Lambda_{\mu\sigma\kappa\lambda}$, hence there will be 20 quantities involving second derivatives of the metric tensor, that cannot be made to vanish at a point by a coordinate transformation.

In a frame where the first derivatives of the metric tensor can be chosen to vanish at a point, the Christoffel symbols also vanish at that point, hence
\[
\frac{d^2x'_{\mu}}{d^2\tau} = 0. \tag{19.10}
\]
The new coordinates at that point are freely falling, hence the name geodesic.

We have spoken before of parallel transport, and concluded that when a vector $A_{\mu}$ is transported an amount $\delta x^\kappa$ parallel to itself, the change in $A_{\mu}$, arising from the change in the coordinates, is
\[
\delta A_{\mu} = \frac{\partial}{\partial x^\kappa} \delta x^\kappa. \tag{19.11}
\]
We can think of the covariant derivative as the difference between the ordinary derivative and the change that would occur if the vector were merely parallel-transported; hence the change in a contravariant vector under parallel transport is
\[
\delta A^\mu = -\frac{\partial}{\partial x^\kappa} \delta x^\kappa. \tag{19.12}
\]
Finally, we note that the 4-velocity $U^\mu = \frac{df}{d\tau}$ is always parallel-transported; moreover, the contraction of the velocity and $A_{\mu}$ is a scalar that is invariant under parallel transport
\[
\delta \left( U^\mu A_{\mu} \right) = \left[ \frac{df}{d\tau} A_{\mu} - A_{\mu} \frac{df}{d\tau} U^\sigma \right] \delta x^\kappa = 0. \tag{19.13}
\]
To put it another way, the angle between a vector that is parallel-transported and the velocity is always constant.
The curvature tensor

Let us parallel-transport a vector around an infinitesimal closed curve parameterized by \( x^\mu(p) \): if \( \tilde{V}_\mu(0) \equiv V_\mu(x^\kappa(0)) \), we find

\[
\int_0^p dp' \frac{d\tilde{V}_\mu(p')}{dp'} = \int_0^p dp' \left\{ \sigma_{\kappa\mu}(p') \tilde{V}_\sigma(p') \frac{dx^\kappa}{dp'} \right\}. \tag{19.14}
\]

Let \( x^\mu(p=0) = \bar{a}^\mu \); we can then expand in Taylor's series about \( p=0 \):

\[
\tilde{V}_\mu(p') \approx V_\mu(a) + \left\{ \sigma_{\kappa\sigma}(a) \right\} (x^\kappa(p') - \bar{a}^\kappa) \tag{19.15}
\]

and

\[
\left\{ \sigma_{\kappa\sigma}(x(p')) \right\} \approx \left\{ \sigma_{\kappa\sigma}(a) \right\} + \left( x^\lambda(p') - \bar{a}^\lambda \right) \frac{\partial}{\partial x^\lambda} \left\{ \sigma_{\kappa\sigma}(a) \right\}. \tag{19.16}
\]

Therefore

\[
\tilde{V}_\mu(p') - \tilde{V}_\mu(a) \approx \int_0^p dp' \left\{ \sigma_{\mu\nu}(a) \right\} + \left( x^\lambda(p') - \bar{a}^\lambda \right) \partial_\lambda \left\{ \sigma_{\mu\nu}(a) \right\}. \tag{19.17}
\]

Thus, expanding in powers of

\[
\delta x^\lambda(p') = x^\lambda(p') - \bar{a}^\lambda,
\]

we find

\[
\tilde{V}_\mu(p) - V_\mu(a) \approx \left\{ \sigma_{\nu\mu}(a) \right\} V_\sigma(a) \int_0^p dp' \frac{dx^\nu}{dp'} + \\
+ \left[ \left\{ \sigma_{\nu\mu}(a) \right\} \left\{ \alpha_{\lambda\sigma}(a) \right\} V_\alpha(a) \right] \partial_\lambda \left\{ \sigma_{\nu\mu}(a) \right\} \int_0^p dp' \frac{dx^\nu}{dp'} \delta x^\lambda(p') + O(\delta x^\nu \delta x^\lambda) \tag{19.18}
\]

We drop the \( O(\delta x^\nu \delta x^\lambda) \) term in Eq. 19.18 because we shall ultimately take the neighborhood of \( a^\lambda \) to be arbitrarily small.

Now, since the parameterization describes a closed curve, we have

\[
\int_0^p dp' \frac{dx^\nu}{dp'} = \oint dx^\nu(p') \tag{19.19}
\]

Thus we have to evaluate
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Lecture 19: The curvature tensor

\[ \oint dx^\nu (p') (x^\lambda (p') - a^\lambda) = \oint dx^\nu (p') x^\lambda (p') \]  
(19.20)

We see immediately that, because \( d(x^\nu x^\lambda) = x^\lambda dx^\nu + x^\nu dx^\lambda \) is a perfect derivative,

\[ \oint dx^\nu x^\lambda = -\oint dx^\lambda x^\nu \]  
(19.21)

Thus we have an expression of the form

\[ \delta V_\mu = \frac{1}{2} R^\sigma_{\mu\nu\lambda} (a) V_\sigma (a) \oint dx^\nu x^\lambda \]  
(19.22)

where equation 19.22 defines the Riemann curvature tensor:

\[ R^\sigma_{\mu\nu\lambda} = \partial_\lambda \left\{ \sigma_\nu \right\}_{\mu\mu} - \partial_\nu \left\{ \sigma_\lambda \right\}_{\mu\mu} + \alpha_\lambda \left\{ \sigma_\nu \right\}_{\mu\mu} - \alpha_\nu \left\{ \sigma_\lambda \right\}_{\mu\mu} = \]  
(19.23)

Clearly, \( R^\sigma_{\mu\nu\lambda} = 0 \) if the space is flat, i.e. if the first and second derivatives of the metric tensor vanish, since then the Christoffel symbols and their first derivatives vanish.

Does this mean that in a freely falling system the curvature tensor is zero? No, because while the Christoffel symbols vanish, their (ordinary) derivatives will not. Thus we can, in principle, distinguish between a flat space and a freely falling system in a curved space, by the non-vanishing of the curvature in the latter case.

We note that \( R^\sigma_{\mu\nu\lambda} \) is a tensor by construction, since everything else in Eq. 19.22 is a tensor. We also note that it is antisymmetric in \( \lambda \nu \). If one lowers the top index to produce the 4th rank covariant tensor

\[ R_{\kappa\mu\nu\lambda} = g_{\kappa\sigma} R^\sigma_{\mu\nu\lambda}, \]

we find the latter satisfies four identities:

\[ R_{\kappa\sigma \mu \nu} \equiv -R_{\sigma \kappa \mu \nu} \]
\[ R_{\kappa\sigma \mu \nu} \equiv -R_{\sigma \kappa \nu \mu} \]
\[ R_{\kappa\sigma \mu \nu} \equiv R_{\mu \nu \kappa \sigma} \]
\[ R_{\kappa\sigma \mu \nu} + R_{\kappa\mu \nu \sigma} + R_{\kappa\nu \sigma \mu} \equiv 0. \]

By virtue of these it is possible to show that only 20 of the components of the tensor \( R^\sigma_{\mu\nu\lambda} \) are independent (see Ohanian and Ruffini, p. 334ff), hence they may be identified with the 20 non-trivial components of the second derivative of the metric tensor. In fact, up to a constant multiplier \( R^\sigma_{\mu\nu\lambda} \) is unique.
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The curvature tensor