The curvature tensor

Covariant derivative of contravariant vector

The covariant derivative of a (contravariant) vector is

$$V^{\mu}{}_{;\nu} = \partial_{\nu} V^{\mu} + \begin{Bmatrix} \mu \\ \nu\sigma \end{Bmatrix} V^{\sigma} . \qquad (19.1)$$

We used this in Eq. 18.2 without explaining it. Where does it come from? We know that the derivative of a scalar is a covariant vector,

df

$$\varphi_{,\mu} = \partial_{\mu} \varphi .$$

Now, suppose the scalar is the contraction of 2 vectors:

$$\varphi = A_{\mu} B^{\mu} \qquad (19.2)$$

then by definition (and the product rule)

$$\partial_{\nu} \varphi = A_{\mu,\nu} B^{\mu} + A_{\mu} B^{\mu}{}_{,\nu} \equiv A_{\mu;\nu} B^{\mu} + A_{\mu} B^{\mu}{}_{;\nu}$$

$$= \left[A_{\mu;\nu} + \begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix} A_{\sigma} \right] B^{\mu} + A_{\mu} B^{\mu}{}_{,\nu} \qquad (19.3)$$

From Eq. 19.3 we have

$$A_{\mu;\nu} B^{\mu} + A_{\mu} B^{\mu}{}_{;\nu} = A_{\mu;\nu} B^{\mu} + \begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix} A_{\sigma} B^{\mu} + A_{\mu} B^{\mu}{}_{,\nu}$$

or

$$A_{\mu} B^{\mu}{}_{;\nu} = A_{\mu} B^{\mu}{}_{,\nu} + \begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix} A_{\sigma} B^{\mu} = A_{\mu} \left[B^{\mu}{}_{,\nu} + \begin{Bmatrix} \mu \\ \nu\sigma \end{Bmatrix} B^{\sigma} \right] \qquad (19.4)$$

and since A_{μ} is arbitrary, we may say

$$B^{\mu}{}_{;\nu} = B^{\mu}{}_{,\nu} + \begin{Bmatrix} \mu \\ \nu\sigma \end{Bmatrix} B^{\sigma} . \qquad (19.5)$$

Covariant derivative of tensor

By similar manipulations, we can identify the covariant derivative of a contravarient second-rank tensor—we write

$$\varphi = A_{\mu} B_{\nu} T^{\mu\nu} \qquad (19.6)$$

and use the product rule again to write

$$\partial_{\kappa} \varphi = A_{\mu,\kappa} B_{\nu} T^{\mu\nu} + A_{\mu} B_{\nu,\kappa} T^{\mu\nu} + A_{\mu} B_{\nu} T^{\mu\nu}{}_{,\kappa}$$

$$\equiv A_{\mu;\kappa} B_{\nu} T^{\mu\nu} + A_{\mu} B_{\nu;\kappa} T^{\mu\nu} + A_{\mu} B_{\nu} T^{\mu\nu}{}_{;\kappa} \qquad (19.7)$$

to find

$$T^{\mu\nu}{}_{;\kappa} = T^{\mu\nu}{}_{,\kappa} + \begin{Bmatrix}\mu \\ \kappa\sigma\end{Bmatrix} T^{\sigma\nu} + \begin{Bmatrix}\nu \\ \kappa\sigma\end{Bmatrix} T^{\mu\sigma} \tag{19.8}$$

and so forth.

Geodesic coordinates

Suppose we locally change coordinates to a system $x'^{\mu} = b^{\mu}{}_{\sigma} x^{\sigma}$, with (linear) transformation coefficients $b^{\mu}{}_{\sigma}$ chosen such that the new metric at that point is

$$g'^{\mu\nu} \stackrel{df}{=} g^{\alpha\beta} b^{\mu}{}_{\alpha} b^{\nu}{}_{\beta} = \eta^{\mu\nu} . \tag{19.9}$$

Equations 19.9 constitute 10 inhomogeneous equations for 16 unknowns $b^{\mu}{}_{\sigma}$, whose determinant,

$$\det\left[g^{\alpha\beta}\right] = \frac{1}{-g} ,$$

is non-zero. Therefore they can always be solved, leaving 6 free parameters. These are in fact the 6 parameters of the Lorentz transformation (3 boost, 3 rotation) which, as we already know, leave the Minkoski metric unchanged.

Moreover, we can specify the coordinates further so that in the new system, all first derivatives of the new metric, $g'_{\mu\nu,\kappa}$ vanish at the point a^{σ}. The coordinates that do this are

$$x'^{\mu} = b^{\mu}{}_{\sigma} x^{\sigma} + \frac{1}{2} \Gamma^{\mu}{}_{\sigma\kappa} \left(x^{\sigma} - a^{\sigma}\right)\left(x^{\kappa} - a^{\kappa}\right) +$$

$$+ \frac{1}{3!} \Lambda^{\mu}{}_{\sigma\kappa\lambda} \left(x^{\sigma} - a^{\sigma}\right)\left(x^{\kappa} - a^{\kappa}\right)\left(x^{\lambda} - a^{\lambda}\right) + \ldots$$

where the coefficients $\Gamma^{\mu}{}_{\sigma\kappa}$ and $\Lambda^{\mu}{}_{\sigma\kappa\lambda}$ are manifestly symmetric in their lower indices, hence represent 20 and 80 independent parameters, respectively.

Problem:
An object with 3 indices that run from 0 to 3 obviously has 64 components. Show that if the object is fully symmetric in the 3 indices, then there are but 20 independent components.
Hence show that $\Lambda^{\mu}{}_{\sigma\kappa\lambda}$ has 80 independent components.

Problem:
Find the relation between the coefficients $\Gamma^{\mu}{}_{\sigma\kappa}$ and the Christoffel symbols $\begin{Bmatrix}\mu \\ \sigma\kappa\end{Bmatrix}(a)$ defined at the point a^{σ} in terms of the (derivatives of the) old metric $g_{\mu\nu}$.

Since there are 20 first derivatives of the metric tensor, we can obviously choose the coefficients $\Gamma^{\mu}{}_{\sigma\kappa}$ to set the derivatives of the new metric tensor equal to zero at one point. Since then the

Christoffel symbol vanishes, the motion of a test body in these new coordinates is unaccelerated, *i.e.* freely falling.

What about the second derivatives of the metric tensor? There are 100 distinct components,

$$g'_{\mu\nu\,,\,\kappa\sigma}\ ,$$

(two distinct pairs of symmetric indices, giving $10 \times 10 = 100$), but only 80 independent parameters $\Lambda^{\mu}{}_{\sigma\kappa\lambda}$, hence there will be 20 quantities involving second derivatives of the metric tensor, that cannot be made to vanish at a point by a coordinate transformation.

In a frame where the first derivatives of the metric tensor can be chosen to vanish at a point, the Christoffel symbols also vanish at that point, hence

$$\frac{d^2 x'^{\mu}}{d^2\tau} = 0\,. \tag{19.10}$$

The new coordinates at that point are freely falling, hence the name *geodesic*.

We have spoken before of parallel transport, and concluded that when a vector A_μ is transported an amount δx^κ parallel to itself, the change in A_μ, arising from the change in the coordinates, is

$$\delta A_\mu = \begin{Bmatrix} \sigma \\ \kappa\mu \end{Bmatrix} A_\sigma\, \delta x^\kappa\,. \tag{19.11}$$

We can think of the covariant derivative as the difference between the ordinary derivative and the change that would occur if the vector were merely parallel-transported; hence the change in a *contravariant* vector under parallel transport is

$$\delta A^\mu = -\begin{Bmatrix} \mu \\ \kappa\sigma \end{Bmatrix} A_\sigma\, \delta x^\kappa\,. \tag{19.12}$$

Finally, we note that the 4-velocity $U^\mu \stackrel{df}{=} \dfrac{dx^\mu}{d\tau}$ is always parallel-transported; moreover, the contraction of the velocity and A_μ is a scalar that is invariant under parallel transport

$$\delta\left(U^\mu A_\mu\right) = \left[U^\sigma \begin{Bmatrix} \mu \\ \sigma\kappa \end{Bmatrix} A_\mu \;-\; A_\mu \begin{Bmatrix} \mu \\ \sigma\kappa \end{Bmatrix} U^\sigma\right] \delta x^\kappa = 0\,. \tag{19.13}$$

To put it another way, the angle between a vector that is parallel-transported and the velocity is always constant.

The curvature tensor

Let us parallel-transport a vector around an infinitesimal closed curve parameterized by $x^{\mu}(p)$: if $\widetilde{V}_{\mu}(0) \equiv V_{\mu}\left(x^{\kappa}(0)\right)$, we find

$$\widetilde{V}_{\mu}(p) - \widetilde{V}_{\mu}(0) = \int_0^p dp' \frac{d\widetilde{V}_{\mu}(p')}{dp'} = \int_0^p dp' \begin{Bmatrix} \sigma \\ \kappa\mu \end{Bmatrix}(p') \widetilde{V}_{\sigma}(p') \frac{dx^{\kappa}}{dp'} . \qquad (19.14)$$

Let $x^{\mu}(p=0) = a^{\mu}$; we can then expand in Taylor's series about $p=0$:

$$\widetilde{V}_{\mu}(p') \approx V_{\mu}(a) + \begin{Bmatrix} \sigma \\ \kappa\sigma \end{Bmatrix}(a) V_{\sigma}(a) \left(x^{\kappa}(p') - a^{\kappa}\right) \qquad (19.15)$$

and

$$\begin{Bmatrix} \sigma \\ \kappa\sigma \end{Bmatrix}(x(p')) \approx \begin{Bmatrix} \sigma \\ \kappa\sigma \end{Bmatrix}(a) + \left(x^{\lambda}(p') - a^{\lambda}\right) \frac{\partial}{\partial x^{\lambda}} \begin{Bmatrix} \sigma \\ \kappa\sigma \end{Bmatrix}(a) . \qquad (19.16)$$

Therefore

$$\tilde{V}_{\mu}(p') - \tilde{V}_{\mu}(a) \approx \int_0^p dp' \left[\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}(a) + \left(x^{\lambda}(p') - a^{\lambda}\right) \partial_{\lambda} \begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}(a) \right] \bullet \left[V_{\sigma}(a) + \left(x^{\kappa}(p') - a^{\kappa}\right) \partial_{\kappa} \begin{Bmatrix} \alpha \\ \kappa\sigma \end{Bmatrix}(a) V_{\alpha}(a) \right] \qquad (19.17)$$

Thus, expanding in powers of

$$\delta x^{\lambda}(p') = x^{\lambda}(p') - a^{\lambda} ,$$

we find

$$\widetilde{V}_{\mu}(p) - V_{\mu}(a) \approx \begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix}(a) V_{\sigma}(a) \int_0^p dp' \frac{dx^{\nu}}{dp'} + \qquad (19.18)$$

$$+ \left[\begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix}(a) \begin{Bmatrix} \alpha \\ \lambda\sigma \end{Bmatrix}(a) V_{\alpha}(a) + V_{\sigma}(a) \partial_{\lambda} \begin{Bmatrix} \sigma \\ \nu\mu \end{Bmatrix}(a) \right] \int_0^p dp' \frac{dx^{\nu}}{dp'} \delta x^{\lambda}(p') + O(\delta x^{\nu} \delta x^{\lambda})$$

We drop the $O(\delta x^{\nu} \delta x^{\lambda})$ term in Eq. 19.18 because we shall ultimately take the neighborhood of a^{λ} to be arbitrarily small.

Now, since the parameterization describes a closed curve, we have

$$\int_0^p dp' \frac{dx^{\nu}}{dp'} = \oint dx^{\nu}(p') \qquad (19.19)$$

Thus we have to evaluate

$$\oint dx^{\nu}(p')\left(x^{\lambda}(p')-a^{\lambda}\right) \equiv \oint dx^{\nu}(p')\,x^{\lambda}(p') \qquad (19.20)$$

We see immediately that, because $d(x^{\nu}x^{\lambda}) = x^{\lambda}dx^{\nu} + x^{\nu}dx^{\lambda}$ is a perfect derivative,

$$\oint dx^{\nu}x^{\lambda} = -\oint dx^{\lambda}x^{\nu} \qquad (19.21)$$

Thus we have an expression of the form

$$\delta\tilde{V}_{\mu} = \frac{1}{2}R^{\sigma}_{\mu\nu\lambda}(a)\,V_{\sigma}(a)\oint dx^{\nu}x^{\lambda} \qquad (19.22)$$

where equation 19.22 defines the ***Riemann curvature tensor***:

$$R^{\sigma}_{\mu\nu\lambda} = \partial_{\lambda}\begin{Bmatrix}\sigma\\ \nu\mu\end{Bmatrix} - \partial_{\nu}\begin{Bmatrix}\sigma\\ \lambda\mu\end{Bmatrix} + \begin{Bmatrix}\alpha\\ \nu\mu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \lambda\alpha\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \lambda\mu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \nu\alpha\end{Bmatrix} \qquad (19.23)$$

Clearly, $R^{\sigma}_{\mu\nu\lambda} = 0$ if the space is flat, *i.e.* if the first and second derivatives of the metric tensor vanish, since then the Christoffel symbols and their first derivatives vanish.

Does this mean that in a freely falling system the curvature tensor is zero? No, because while the Christoffel symbols vanish, their (ordinary) derivatives will not. Thus we can, in principle, distinguish between a flat space and a freely falling system in a curved space, by the non-vanishing of the curvature in the latter case.

We note that $R^{\sigma}_{\mu\nu\lambda}$ is a tensor by construction, since everything else in Eq. 19.22 is a tensor. We also note that it is antisymmetric in $\lambda\nu$. If one lowers the top index to produce the 4th rank covariant tensor

$$R_{\kappa\mu\,\nu\lambda} = g_{\kappa\sigma}R^{\sigma}_{\mu\nu\lambda}\,,$$

we find the latter satisfies four identities:

$$R_{\kappa\sigma\,\mu\nu} \equiv -R_{\sigma\kappa\,\mu\nu}$$
$$R_{\kappa\sigma\,\mu\nu} \equiv -R_{\kappa\sigma\,\nu\mu}$$
$$R_{\kappa\sigma\,\mu\nu} \equiv R_{\mu\nu\,\kappa\sigma}$$
$$R_{\kappa\sigma\,\mu\nu} + R_{\kappa\mu\,\nu\sigma} + R_{\kappa\nu\,\sigma\mu} \equiv 0\ .$$

By virtue of these it is possible to show that only 20 of the components of the tensor $R^{\sigma}_{\mu\nu\lambda}$ are independent (see Ohanian and Ruffini, p. 334ff), hence they may be identified with the 20 non-trivial components of the second derivative of the metric tensor. In fact, up to a constant multiplier $R^{\sigma}_{\mu\nu\lambda}$ is unique.