The Einstein equations

Properties of the curvature tensor

In §19 we defined the Riemann curvature tensor

$$R^{\sigma}_{\mu\nu\lambda} = \partial_\lambda \begin{Bmatrix}\sigma\\ \nu\mu\end{Bmatrix} - \partial_\nu \begin{Bmatrix}\sigma\\ \lambda\mu\end{Bmatrix} + \begin{Bmatrix}\alpha\\ \nu\mu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \lambda\alpha\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \lambda\mu\end{Bmatrix}\begin{Bmatrix}\sigma\\ \nu\alpha\end{Bmatrix} \qquad (19.23\ )$$

by considering the change of a vector that is parallel-transported around a closed curve. We can also consider the second covariant derivative of a vector:

$$A_{\mu\,;\,\nu\,;\,\lambda} = A_{\mu\,,\,\nu\,,\,\lambda} - A_\beta\,\partial_\lambda \begin{Bmatrix}\beta\\ \mu\nu\end{Bmatrix} - \begin{Bmatrix}\beta\\ \mu\nu\end{Bmatrix} A_{\beta\,,\,\lambda} - \begin{Bmatrix}\beta\\ \mu\lambda\end{Bmatrix} A_{\beta\,;\,\nu} - \begin{Bmatrix}\beta\\ \nu\lambda\end{Bmatrix} A_{\mu\,;\,\beta} \qquad (20.1)$$

Now consider $A_{\mu\,;\,\nu\,;\,\lambda} - A_{\mu\,;\,\lambda\,;\,\nu}$: on eliminating terms symmetric in $\nu\lambda$† we find

$$A_{\mu\,;\,\nu\,;\,\lambda} - A_{\mu\,;\,\lambda\,;\,\nu} = R^{\beta}_{\mu\nu\lambda}\,A_\beta\,. \qquad (20.2)$$

What are the properties of $R^{\beta}_{\mu\nu\lambda}$? First, it is a tensor. If the previous derivation (in §19) was unconvincing, Eq. 20.2 should fix that: the left hand side is clearly a tensor (the difference of two tensors at a point is a tensor), so the right hand side is also a tensor. But A_β is an arbitrary vector field, hence $R^{\beta}_{\mu\nu\lambda}$ is a tensor.

Next, let us bring down the contravariant index with $g_{\alpha\beta}$:

$$\begin{aligned} R_{\alpha\beta\,\mu\nu} &= g_{\alpha\lambda}\,R^{\lambda}_{\beta\mu\nu} \\ &= [\beta\nu\,,\,\alpha]_{,\,\mu} - [\beta\mu\,,\,\alpha]_{,\,\nu} + [\beta\nu\,,\,\lambda]\,g_{\alpha\sigma}\,g^{\sigma\lambda}{}_{,\,\mu} + \\ &\qquad + [\beta\nu\,,\,\lambda]\,g_{\alpha\sigma}\,g^{\sigma\lambda}{}_{,\,\mu} - [\beta\mu\,,\,\lambda]\,g_{\alpha\sigma}\,g^{\sigma\lambda}{}_{,\,\nu} + \\ &\qquad + \Big([\beta\nu\,,\,\sigma]\,[\rho\mu\,,\,\alpha] - [\beta\mu\,,\,\sigma]\,[\rho\nu\,,\,\alpha]\Big)g^{\rho\sigma} \end{aligned}$$

which, as we easily see, has the (anti)symmetry property:

$$R_{\alpha\beta\,\mu\nu} = -\,R_{\alpha\beta\,\nu\mu}\,. \qquad (20.3)$$

It is possible---but not easy!---to see that

$$R_{\alpha\beta\,\mu\nu} = -\,R_{\beta\alpha\,\mu\nu} \qquad (20.4)$$

and that

$$R_{\alpha\beta\,\mu\nu} = R_{\mu\nu\,\alpha\beta}\,. \qquad (20.5)$$

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† …bearing in mind that ordinary partial derivatives commute.

Gravitation and Cosmology

Some definitions:

A final symmetry with regard to index permutations is

$$R_{\alpha\beta\ \mu\nu} + R_{\alpha\mu\ \nu\beta} + R_{\alpha\nu\ \beta\mu} \equiv 0\,. \qquad (20.6)$$

We note that the symmetry 20.6 really constitutes only one relation for the curvature tensor, since unless the indices are all different, it reduces to 20.3, 20.4 or 20.5. This leads to the previously noted fact that only 20 components of the curvature tensor are independent. We can see this by realizing that, because there are only 6 possible pairs of indices $\alpha\beta$ or $\mu\nu$, the curvature tensor is by virtue of 20.5 like a 6×6 symmetric matrix, which has 21 independent components. However, the additional relation 20.6 reduces the number of components from 21 to 20.

Finally, we also have the Bianchi identity:

$$R_{\alpha\beta\,\mu\nu\,;\,\sigma} + R_{\alpha\beta\,\nu\sigma\,;\,\mu} + R_{\alpha\beta\,\sigma\mu\,;\,\nu} = 0\,, \qquad (20.7)$$

which is far from easy to prove. The easiest method is the trick in Ohanian and Ruffini, of working in a geodesic coordinate system. Then the Christoffel symbols vanish and all covariant derivatives reduce to ordinary partial derivatives which commute. But since the identity is true in one coordinate system it must (*via* general covariance) be tru in all.

Some definitions:

The Ricci tensor is defined as the contraction of one of the first pair of indices with one of the second pair:

$$R_{\beta\nu} \stackrel{df}{=} g^{\alpha\mu} R_{\alpha\beta\mu\nu}\,;$$

the curvature scalar is:

$$R \stackrel{df}{=} g^{\beta\nu} R_{\beta\nu} \equiv g^{\beta\nu} g^{\alpha\mu} R_{\alpha\beta\mu\nu}$$

The antisymmetry relations 20.3-20.6, plus the Bianchi identity 20.7 give the second Bianchi identity

$$\left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right]_{;\,\sigma} = 0\,. \qquad (20.8)$$

Gravitation

As we have seen, the Einstein theory of gravitation has the form

$$\Phi^{\mu\nu} = -8\pi G\, T^{\mu\nu} \qquad (20.9)$$

where $T^{\mu\nu}$ is the energy-momentum tensor of matter, and $\Phi^{\mu\nu}$ is constructed from gravitational fields $h^{\mu\nu}$. We obtained the equation in Minkowski space by insisting on two criteria, namely that the field equations contain up to second derivatives, that it respect energy-momentum conservation, and that it obey the principle of equivalence. The form of our equations, together with thought experiments in accelerated frames, has led us to equate gravitation with geometry. Thus, we should conjecture that the field equations must have the generally covariant form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda\, g^{\mu\nu} = -8\pi G\, T^{\mu\nu} \qquad (20.10\,)$$

where Λ is an (unknown) constant, possibly zero, called the *cosmological constant*.

Loose ends

In adding in a term $\Lambda\, g^{\mu\nu}$ involving the cosmological constant Λ, we assumed that

$$g^{\mu\nu}{}_{;\,\sigma} = 0 \,.$$

In fact, it is easier to show that

$$g_{\mu\nu\,;\,\sigma} = 0 \;:$$

we have

$$\begin{aligned} g_{\mu\nu\,;\,\sigma} &= g_{\mu\nu\,,\,\sigma} - \begin{Bmatrix} \beta \\ \nu\sigma \end{Bmatrix} g_{\mu\beta} - \begin{Bmatrix} \beta \\ \mu\sigma \end{Bmatrix} g_{\beta\nu} \\ &\equiv g_{\mu\nu\,,\,\sigma} - [\mu\sigma\,,\nu] - [\nu\sigma\,,\mu] \qquad (20.11) \\ &= g_{\mu\nu\,,\,\sigma} - \tfrac{1}{2}\Big[g_{\sigma\nu\,,\,\mu} + g_{\mu\nu\,,\,\sigma} - g_{\mu\sigma\,,\,\nu} + g_{\nu\mu\,,\,\sigma} + g_{\sigma\mu\,,\,\nu} - g_{\sigma\nu\,,\,\mu}\Big] = 0 \,. \end{aligned}$$

We also note that since $g^{\mu\kappa}\, g_{\kappa\nu} = \delta^{\mu}{}_{\nu}$,

$$\Big[g^{\mu\kappa}\, g_{\kappa\nu}\Big]_{;\,\sigma} = g^{\mu\kappa}{}_{;\,\sigma}\; g_{\kappa\nu} + g^{\mu\kappa}\, g_{\kappa\nu\,;\,\sigma} = \Big[\delta^{\mu}{}_{\nu}\Big]_{;\,\sigma} = 0 \qquad (20.12)$$

and since $g_{\kappa\nu\,;\,\sigma} = 0$, then $g^{\mu\kappa}{}_{;\,\sigma} = 0$. QED.

Cosmological constant

We noted that the Einstein equations could in principle contain a cosmological term $\Lambda\, g^{\mu\nu}$. It is easy to see that in the absence of matter, since $g_{00} \approx 1 + 2\Phi$ (Φ is the dimensionless Newtonian potential) the equations look something like

$$\nabla^2 \Phi = -\Lambda \,. \qquad (20.13)$$

The solution of this—finite at the origin—is $\Phi \approx -\frac{1}{6}\Lambda r^2$. How can we tell, experimentally, whether such a term is present or not? Consider a circular orbit about the Sun. The balance between centrifugal and gravitational force gives

$$\omega^2 = \frac{GM}{r^3} + \frac{1}{3}\Lambda \,. \qquad (20.14)$$

Thus, by plotting the square of angular velocity *vs.* the inverse-cube of orbital radius for all the planets, we can determine whether there is a non-zero intercept.

Suppose we do this with the planets Venus-Neptune (low eccentricities!):

Orbital Data for the Solar System

Planet	Period (yr)	Orbital mean radius (10^6 km)
Venus	0.61521	108.2
Earth	1.00004	149.54
Mars	1.88089	225.95
Jupiter	11.86223	776.5

Orbital Data for the Solar System

Planet	Period (yr)	Orbital mean radius (10^6 km)
Saturn	29.45772	1423
Uranus	84.013	2863
Neptune	164.79	4498

The equation

$$T^{-2} = a + bR^{-3} \tag{20.15}$$

used to fit the above data yields least-squares best-fit parameters

$$a = -1.3655 \times 10^{-3}$$

$$b = 3.3478 \times 10^{6}$$

Problem:
Is the value of b reasonable? What are its units?

We conclude from the intercept *a* above that

$$|\Lambda| \le 1.6{\times}10^{-16}\ \mathrm{sec}^{-2}.$$

From the behavior of galaxies and clusters we can----similarly----set a bound

$$|\Lambda| \le 1.6{\times}10^{-33}\ \mathrm{sec}^{-2}.$$

Is this large or small? If we take the density of mass-energy in the universe to be the Einstein critical density 10^{-29} gm/cm^3 , we have

$$8\pi G\rho_{crit} \approx 1.7{\times}10^{-35}\ \mathrm{sec}^{-2}$$

which is 2 orders of magnitude smaller than the best astronomical bound on Λ. That is, a cosmological constant this large would absolutely dominate cosmology, and the matter would have nothing to do with anything.

Further reading:

S. Weinberg, “The cosmological constant problem”, *Rev. Mod. Phys.* **61** (1989) 1.

The Schwarzschild solution

We are now going to seek solutions of the Einstein equations (*sans* cosmological term)

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G\, T^{\mu\nu} \tag{20.16}$$

By the way, we never mentioned how we got the overall constant in front of $T^{\mu\nu}$: this came from demanding that Eq. 20.16 reduce to the Newtonian gravitational potential in the weak-field limit, with the identification

$$g_{00} \approx 1 + 2\Phi\,. \tag{20.17}$$

Now the object is to seek a solution that is spherically symmetric and time-independent (physical insight tells us this should be the case), in which the metric takes the form

$$(d\tau)^2 \stackrel{df}{=} g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$= B(r)\,(dt)^2 - A(r)\,(dr)^2 - r^2\left[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right]. \qquad (20.18)$$

That is,

$$\begin{aligned} g_{tt} &= B(r) \\ g_{rr} &= A(r) \\ g_{\theta\theta} &= r^2 \\ g_{\varphi\varphi} &= r^2 \sin^2\theta \end{aligned} \qquad (20.19)$$

with the off-diagonal elements zero.

The determinant of the metric tensor is then (we shall need $\sqrt{g}$ in volume elements)

$$g = -\det\left(g_{\mu\nu}\right) = A(r)\,B(r)\,r^4 \sin^2\theta\,. \qquad (20.20)$$