The Schwarzschild solution

The field of a point mass

We now seek solutions of the Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \tag{21.1}$$

where the source is an isolated point mass, ***M***. By locating the mass at the origin of coordinates, we may be certain the solution will be rotationally invariant. This symmetry will reduce considerably the labor of finding the solution.

The most general rotationally invariant quantities in 3-space are

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$

$$\vec{x} \cdot d\vec{x} = x\,dx + y\,dy + z\,dz \tag{21.2}$$

$$d\vec{x} \cdot d\vec{x} = (dx)^2 + (dy)^2 + (dz)^2$$

so we may write the most general proper time interval (with no explicit time-dependence)

$$(d\tau)^2 = F(r)\,(dt)^2 - 2E(r)\,dt\,(\vec{x} \cdot d\vec{x}) - D(r)\,(\vec{x} \cdot d\vec{x})^2 - C(r)\,(d\vec{x} \cdot d\vec{x})\,. \tag{21.3}$$

Working in spherical polar coordinates

$$x = r\cos\varphi\sin\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\theta$$

we can rewrite this as

$$(d\tau)^2 = F(r)\,(dt)^2 - 2rE(r)\,dt\,dr - r^2 D(r)\,(dr)^2 -$$
$$- C(r)\left[(dr)^2 + r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2\right]. \tag{21.4}$$

By suitable changes of coordinate and suitable redefinitions (see Weinberg, *e.g.*), we can rewrite the proper time in the form

$$(d\tau)^2 = B(r)\,(dt)^2 - A(r)\,(dr)^2 - r^2\,(d\theta)^2 - r^2\sin^2\theta\,(d\varphi)^2 \tag{21.5}$$

That is, the metric becomes

$$g_{tt} = B(r) \qquad g_{\theta\theta} = -r^2$$
$$g_{rr} = -A(r) \qquad g_{\varphi\varphi} = -r^2\sin^2\theta \tag{21.6}$$

Clearly, since the metric is diagonal,

$$g^{tt} = \frac{1}{B(r)} \qquad g^{\theta\theta} = \frac{-1}{r^2}$$
$$g^{rr} = \frac{-1}{A(r)} \qquad g^{\varphi\varphi} = \frac{-1}{r^2 \sin^2\theta} \qquad (21.7)$$

The determinant of $g_{\mu\nu}$ is

$$g = -\det\left(g_{\mu\nu}\right) = A(r)\, B(r)\; r^4 \sin^2\theta\,. \qquad (21.8)$$

We shall need all the terms of the Christoffel symbol

$$\left\{ {\sigma \atop \mu\nu} \right\} = \frac{1}{2} g^{\sigma\lambda} \left[g_{\lambda\mu\,,\,\nu} + g_{\lambda\nu\,,\,\mu} - g_{\mu\nu\,,\,\lambda} \right] \qquad (21.9)$$

to compute the Ricci tensor $R^{\mu\nu}$. Now we enumerate the terms:

$$\left\{ {t \atop t\,t} \right\} = \frac{1}{2} g^{tt} \left[g_{tt\,,\,t} + g_{tt\,,\,t} - g_{tt\,,\,t} \right] = 0 \qquad [1]$$

$$\left\{ {t \atop r\,t} \right\} = \frac{1}{2} g^{tt} \left[g_{tr\,,\,t} + g_{tt\,,\,r} - g_{rt\,,\,t} \right] = \frac{1}{2} \partial_r \ln\left[B(r)\right] \qquad [2]$$

$$\left\{ {t \atop \theta\,t} \right\} = \frac{1}{2} g^{tt} \left[g_{t\theta\,,\,t} + g_{tt\,,\,\theta} - g_{\theta t\,,\,t} \right] = 0 \qquad [3]$$

$$\left\{ {t \atop \varphi\,t} \right\} = \frac{1}{2} g^{tt} \left[g_{t\varphi\,,\,t} + g_{tt\,,\,\varphi} - g_{\varphi t\,,\,t} \right] = 0 \qquad [4]$$

$$\left\{ {t \atop r\,r} \right\} = \frac{1}{2} g^{tt} \left[g_{rt\,,\,r} + g_{tr\,,\,r} - g_{rr\,,\,t} \right] = 0 \qquad [5]$$

$$\left\{ {t \atop \theta\,r} \right\} = \frac{1}{2} g^{tt} \left[g_{t\theta\,,\,r} + g_{tr\,,\,\theta} - g_{\theta\,r\,,\,t} \right] = 0 \qquad [6]$$

$$\left\{ {t \atop \varphi\,r} \right\} = \frac{1}{2} g^{tt} \left[g_{t\varphi\,,\,r} + g_{tr\,,\,\varphi} - g_{\varphi\,r\,,\,t} \right] = 0 \qquad [7]$$

$$\left\{ {t \atop \theta\,\theta} \right\} = \left\{ {t \atop \varphi\,\theta} \right\} = \left\{ {t \atop \varphi\,\varphi} \right\} = 0 \qquad [8\text{-}10]$$

$$\left\{ {r \atop t\,t} \right\} = \frac{1}{2} g^{rr} \left[2g_{rt\,,\,t} - g_{tt\,,\,r} \right] = \frac{B'(r)}{2A(r)} \qquad [11]$$

$$\left\{ {r \atop r\,t} \right\} = \left\{ {r \atop t\,\theta} \right\} = \left\{ {r \atop \varphi\,t} \right\} = 0 \qquad [12\text{-}14]$$

$$\left\{ {r \atop r\,r} \right\} = \frac{1}{2} g^{rr} \left[g_{rr\,,\,r} + g_{rr\,,\,r} - g_{rr\,,\,r} \right] = \frac{1}{2} \partial_r \ln\left[A(r)\right] \qquad [15]$$

$$\left\{ {r \atop r\,\theta} \right\} = \left\{ {r \atop \varphi\,r} \right\} = 0 \qquad [16\text{-}17]$$

$$\begin{Bmatrix} r \\ \theta\,\theta \end{Bmatrix} = \frac{1}{2} g^{rr} \left[2g_{r\theta\,,\,\theta} - g_{\theta\theta\,,\,r} \right] = \frac{-r}{A(r)} \qquad [18]$$

$$\begin{Bmatrix} r \\ \theta\,\varphi \end{Bmatrix} = \frac{1}{2} g^{rr} \left[g_{r\theta\,,\,\varphi} + g_{r\varphi\,,\,\theta} - g_{\theta\varphi\,,\,r} \right] = 0 \qquad [19]$$

$$\begin{Bmatrix} r \\ \varphi\,\varphi \end{Bmatrix} = \frac{1}{2} g^{rr} \left[g_{r\varphi\,,\,\varphi} + g_{r\varphi\,,\,\varphi} - g_{\varphi\varphi\,,\,r} \right] = \frac{-r\sin^2\theta}{A(r)} \qquad [20]$$

$$\begin{Bmatrix} \theta \\ t\,t \end{Bmatrix} = \frac{1}{2} g^{\theta\theta} \left[g_{\theta\,t\,,\,t} + g_{t\theta\,,\,t} - g_{tt\,,\,\theta} \right] = 0 \qquad [21]$$

$$\begin{Bmatrix} \theta \\ r\,t \end{Bmatrix} = \begin{Bmatrix} \theta \\ \theta\,t \end{Bmatrix} = \begin{Bmatrix} \theta \\ \varphi\,t \end{Bmatrix} = 0 \qquad [22\text{-}24]$$

$$\begin{Bmatrix} \theta \\ r\,r \end{Bmatrix} = \frac{1}{2} g^{\theta\theta} \left[g_{\theta\,r\,,\,r} + g_{r\theta\,,\,r} - g_{rr\,,\,\theta} \right] = 0 \qquad [25]$$

$$\begin{Bmatrix} \theta \\ r\,\theta \end{Bmatrix} = \frac{1}{2} g^{\theta\theta} \left[g_{\theta\,r\,,\,\theta} + g_{\theta\theta\,,\,r} - g_{\theta\,r\,,\,\theta} \right] = \frac{1}{r} \qquad [26]$$

$$\begin{Bmatrix} \theta \\ r\,\varphi \end{Bmatrix} = \begin{Bmatrix} \theta \\ \theta\,\theta \end{Bmatrix} = \begin{Bmatrix} \theta \\ \theta\,\varphi \end{Bmatrix} = 0 \qquad [27\text{-}29]$$

$$\begin{Bmatrix} \theta \\ \varphi\,\varphi \end{Bmatrix} = \frac{1}{2} g^{\theta\theta} \left[g_{\theta\,\varphi\,,\,\varphi} + g_{\varphi\,\theta\,,\,\varphi} - g_{\varphi\varphi\,,\,\theta} \right] = -\sin\theta\cos\theta \qquad [30]$$

$$\begin{Bmatrix} \varphi \\ t\,t \end{Bmatrix} = \frac{1}{2} g^{\varphi\varphi} \left[g_{\varphi\,t\,,\,t} + g_{t\,\varphi\,,\,t} - g_{tt\,,\,\varphi} \right] = 0 \qquad [31]$$

$$\begin{Bmatrix} \varphi \\ t\,r \end{Bmatrix} = \begin{Bmatrix} \varphi \\ t\,\theta \end{Bmatrix} = \begin{Bmatrix} \varphi \\ t\,\varphi \end{Bmatrix} = 0 \qquad [32\text{-}34]$$

$$\begin{Bmatrix} \varphi \\ r\,r \end{Bmatrix} = \frac{1}{2} g^{\varphi\varphi} \left[g_{\varphi\,r\,,\,r} + g_{r\,\varphi\,,\,r} - g_{rr\,,\,\varphi} \right] = 0 \qquad [35]$$

$$\begin{Bmatrix} \varphi \\ r\,\theta \end{Bmatrix} = \frac{1}{2} g^{\varphi\varphi} \left[g_{\varphi\,r\,,\,\theta} + g_{\theta\,\varphi\,,\,r} - g_{r\,\theta\,,\,\varphi} \right] = 0 \qquad [36]$$

$$\begin{Bmatrix} \varphi \\ r\,\varphi \end{Bmatrix} = \frac{1}{2} g^{\varphi\varphi} \left[g_{\varphi\,r\,,\,\varphi} + g_{\varphi\,\varphi\,,\,r} - g_{r\,\varphi\,,\,\varphi} \right] = \frac{1}{r} \qquad [37]$$

$$\begin{Bmatrix} \varphi \\ \theta\,\varphi \end{Bmatrix} = \frac{1}{2} g^{\varphi\varphi} \left[g_{\varphi\theta\,,\,\varphi} + g_{\varphi\varphi\,,\,\theta} - g_{\theta\varphi\,,\,\varphi} \right] = \cot\theta \qquad [38]$$

$$\begin{Bmatrix} \varphi \\ \theta\,\theta \end{Bmatrix} = \begin{Bmatrix} \varphi \\ \varphi\,\varphi \end{Bmatrix} = 0 \qquad [39\text{-}40]$$

There are 40 distinct Christoffel symbols, as expected. Next we have 10 combinations to work out for the Ricci tensor

$$R_{\mu\nu} = \partial_\nu \begin{Bmatrix} \lambda \\ \mu\,\lambda \end{Bmatrix} - \partial_\lambda \begin{Bmatrix} \lambda \\ \mu\,\nu \end{Bmatrix} + \begin{Bmatrix} \sigma \\ \mu\,\lambda \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\nu \end{Bmatrix} - \begin{Bmatrix} \sigma \\ \mu\,\nu \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\lambda \end{Bmatrix}. \tag{21.10}$$

Only the 4 diagonal terms are non-vanishing:

$$R_{tt} = \frac{-B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} \tag{21.11t}$$

$$R_{rr} = \frac{-B''}{2B} + \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{A'}{A} \tag{21.11r}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} \tag{21.11θ}$$

$$R_{\varphi\varphi} = \sin^2\theta \;\, R_{\theta\theta} \tag{21.11φ}$$