The Schwarzschild solution, cont'd

We continue to seek solutions of the Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \qquad (21.1)$$

where the source is an isolated point mass, M.

The metric is given by

$$(d\tau)^2 = B(r)\,(dt)^2 - A(r)\,(dr)^2 - r^2\,(d\theta)^2 - r^2 \sin^2\theta\,(d\varphi)^2 \qquad (22.1)$$

hence

$$g = -\det\left(g_{\mu\nu}\right) = A(r)\,B(r)\,r^4 \sin^2\theta\,. \qquad (22.2)$$

We need to express

$$R_{\mu\nu} = \partial_\nu \begin{Bmatrix} \lambda \\ \mu\,\lambda \end{Bmatrix} - \partial_\lambda \begin{Bmatrix} \lambda \\ \mu\,\nu \end{Bmatrix} + \begin{Bmatrix} \sigma \\ \mu\,\lambda \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\nu \end{Bmatrix} - \begin{Bmatrix} \sigma \\ \mu\,\nu \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\lambda \end{Bmatrix} \qquad (22.3)$$

in terms of $A(r)$, $B(r)$, r and θ. Recall that

$$\begin{Bmatrix} \lambda \\ \mu\,\lambda \end{Bmatrix} = \frac{1}{2} \partial_\mu \ln(g)\,; \qquad (22.4)$$

the other derivative term is more easily worked out by components:

$$\partial_\lambda \begin{Bmatrix} \lambda \\ \mu\,\nu \end{Bmatrix} = \partial_t \begin{Bmatrix} t \\ \mu\,\nu \end{Bmatrix} + \partial_r \begin{Bmatrix} r \\ \mu\,\nu \end{Bmatrix} + \partial_\theta \begin{Bmatrix} \theta \\ \mu\,\nu \end{Bmatrix} + \partial_\varphi \begin{Bmatrix} \varphi \\ \mu\,\nu \end{Bmatrix}\,; \qquad (22.5)$$

since everything is time- and φ-independent, the first and fourth terms of Eq. 22.5 vanish. Of the 10 terms $\begin{Bmatrix} \theta \\ \mu\,\nu \end{Bmatrix}$, only two are non-zero, and only one of these depends on θ:

$$\begin{Bmatrix} \theta \\ \varphi\,\varphi \end{Bmatrix} = -\sin\theta\cos\theta\,.$$

All four non-vanishing terms in $\begin{Bmatrix} r \\ \mu\,\nu \end{Bmatrix}$ are diagonal in $\mu\nu$.

Since g is the product of four factors, $\frac{1}{2}\ln(g)$ is the sum of four terms, each depending on either r or θ, but not both. Hence only diagonal elements, rr and $\theta\theta$ can emerge from it.

Next we look at $\begin{Bmatrix} \sigma \\ \mu\,\lambda \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\nu \end{Bmatrix}$; if σ is t, we get diagonal rr and tt elements. If σ is θ, we get rr and $\theta\theta$ elements. If σ is v, we find rr and $\varphi\varphi$ elements.

Considering now $\begin{Bmatrix} \sigma \\ \mu\,\nu \end{Bmatrix} \begin{Bmatrix} \lambda \\ \sigma\,\lambda \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \sigma \\ \mu\,\nu \end{Bmatrix} \partial_\sigma \ln(g)$ we see that σ can only be r or θ. This permits tt, rr, $\theta\theta$ and $\varphi\varphi$ terms. It also permits an $r\theta$ term, but this is cancelled off.

I leave the remainder of the algebraic procedure to the diligent student. It is one of those things worth working out once in one's life, but the pleasure dims with repetition. The results are

$$R_{tt} = \frac{-B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} \qquad (21.11t)$$

$$R_{rr} = \frac{-B''}{2B} + \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{A'}{A} \qquad (21.11r)$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} \qquad (21.11\theta)$$

$$R_{\varphi\varphi} = \sin^2\theta \; R_{\theta\theta} \qquad (21.11\varphi)$$

Now, do we have to evaluate R, the curvature scalar? No, because if we contract Eq. 21.1 we find

$$R - \tfrac{1}{2}\times 4\, R = -8\pi G T \qquad (22.6)$$

or

$$R = 4\pi G T.$$

Thus,

$$R^{\mu\nu} = -8\pi G\left(T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu}\, T\right). \qquad (22.7)$$

Now, outside a point source, $R^{\mu\nu} = 0$. We also impose the boundary condition that the metric becomes Minkowskian as $r \to \infty$. Thus,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1\,. \qquad (22.8)$$

We turn now to actually solving Eq. 21.11. We eliminate the $B''(r)$ between R_{tt} and R_{rr} :

$$\frac{R_{tt}}{B} - \frac{R_{rr}}{A} = \frac{-1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \qquad (22.9)$$

hence

$$\ln A + \ln B = \text{const.}$$

or

$$A(r)\, B(r) = 1\,. \qquad (22.10)$$

(We get Eq. 22.10 by imposing the boundary condition at ∞ .)

Thus we can write

$$R_{\theta\theta} = -1 + rB' + B = 0 \qquad (22.11)$$

$$R_{rr} = \frac{-B''}{2B} - \frac{1}{r}\frac{B'}{B} = 0\,. \qquad (22.12)$$

Since, from Eq. 22.11,

$$[r\,B(r)]' = 1\,, \qquad (22.13)$$

$$rB(r) = r + \text{const.}$$

We know already from the weak field approximation that for large r, $g_{tt} = 1 + 2\,\Phi(r)$, hence by matching we find that

$$B(r) \overset{df}{=} g_{tt}(r) = 1 - \frac{2MG}{r} \tag{22.14}$$

and from Eq. 22.10,

$$A(r) \overset{df}{=} -g_{rr}(r) = \left(1 - \frac{2MG}{r}\right)^{-1} \tag{22.15}$$

Eq. 22.14-15 are known as the *Schwarzschild solution*, or the *Schwarzschild metric*.

Keppler problem in General Relativity

We have just seen that the Schwarzschild metric of a (non-rotating) point mass is

$$(d\tau)^2 = \left(1 - \frac{2MG}{r}\right)(dt)^2 - \left(1 - \frac{2MG}{r}\right)^{-1}(dr)^2 - \\ - r^2\,(d\theta)^2 - r^2\sin^2\theta\,(d\varphi)^2 \,. \tag{22.16}$$

There are several ways to find the equation of motion of a test body in the gravitational field of a large mass. The simplest is the geodesic equation,

$$\frac{d^2 x^\mu}{d\tau^2} + \begin{Bmatrix} \mu \\ \kappa\,\lambda \end{Bmatrix} \frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = 0\,, \qquad (22.217$$

but we could also use Hamilton's principle directly:

$$\delta\int d\tau = \delta\int dp\,\Lambda \equiv \delta\int dp\left(g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{1/2} = 0\,. \tag{22.18}$$

Applied to test motion in a Schwarzschild metric, we have the Euler-Lagrange equations

$$\begin{aligned} &\frac{d}{dp}\left(B(r)\frac{dt}{dp}\frac{1}{\Lambda}\right) = 0 \\ &\frac{d}{dp}\left(r^2\sin^2\theta\,\frac{d\varphi}{dp}\frac{1}{\Lambda}\right) = 0 \\ &\frac{d}{dp}\left(r^2\,\frac{d\theta}{dp}\frac{1}{\Lambda}\right) = r^2\left(\frac{d\varphi}{dp}\right)^2\frac{\sin\theta\cos\theta}{\Lambda} \\ -&\frac{d}{dp}\left(A(r)\frac{dr}{dp}\frac{1}{\Lambda}\right) = \frac{d\Lambda}{dr} \end{aligned} \tag{22.19}$$

Since $\Lambda \overset{df}{=} \dfrac{d\tau}{dp}$, we have from Eq. 22.19

$$B(r)\frac{dt}{d\tau} = \text{const.} \tag{22.20}$$

We choose const. = 1 so that $\tau \to t$ as space-time becomes flat. Next,

$$r^2 \sin^2\theta \frac{d\varphi}{d\tau} = \text{const.} = \frac{J}{m} \tag{22.21}$$

leading to

$$\frac{d}{d\tau}\left(r^2 \frac{d\theta}{d\tau}\right) = \frac{J^2 \cos\theta}{m^2 r^2 \sin^3\theta} \tag{22.22}$$

Equation 22.22 is a second-order differential equation, hence has two constants that can be chosen at $\tau = 0$. With $\theta(0) = \frac{\pi}{2}$ and $\left.\frac{d\theta}{d\tau}\right|_{\tau=0} = 0$, Eq. 22.22 guarantees $\theta(\tau) = \frac{\pi}{2}$ for all subsequent (proper) time.

Now the equation of motion for r may be integrated once using the integrating factor $\frac{dr}{d\tau}$; the result is

$$-\tfrac{1}{2}A(r)\left(\frac{dr}{d\tau}\right)^2 + \tfrac{1}{2}A(r) - \tfrac{1}{2}\frac{J^2}{m^2 r^2} = \text{const.} \tag{22.23}$$

We can now look at the small-field, small-velocity limit:

$$\tfrac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \tfrac{1}{2}\frac{J^2}{m^2 r^2} - \tfrac{1}{2} = \frac{E}{m} - \tfrac{1}{2} \tag{22.24}$$

Eq. 22.24 is just the energy equation derived by integrating Newton's Law once. This guides us to write

$$-A(r)\left(\frac{dr}{d\tau}\right)^2 + A(r) - \frac{J^2}{m^2 r^2} = 1 - \frac{2E}{m}. \tag{22.25}$$

Taking the square root and dividing by $\frac{d\varphi}{d\tau}$, we obtain the General Relativistic analogue of the Newtonian orbital equation:

$$\frac{dr}{d\varphi} = \pm\left[1 - \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2E}{m} + \frac{J^2}{m^2r^2}\right)\right]^{1/2} \frac{mr^2}{J} \tag{22.26}$$

For a bound orbit, we must have $E < 0$.