Solution of the Keppler problem

Perihelion precession

The equation for the orbit of a test particle in the field of a point gravitating mass (Schwarzschild metric) is

$$\frac{dr}{d\varphi} = \pm\left[1 - \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2E}{m} + \frac{J^2}{m^2r^2}\right)\right]^{1/2} \frac{mr^2}{J} \tag{22.26}$$

By changing variables to $u = \frac{1}{r}$, we find an equation of form

$$\frac{du}{d\varphi} = \pm\left[-u^2 + 2au - b + 2MGu^3\right]^{1/2}. \tag{23.1}$$

Clearly the expression within the square root can be factored (since it is cubic, it has 3 roots; moreover, since the number of sign changes among the coefficients is 3, the roots are all real; finally, the roots are all positive since it starts out negative at $u=0$ and becomes positive for large u, whereas for $u<0$ it is negative definite), giving

$$-u^2 + 2au - b + 2MGu^3 = \left(u_> - u\right)\left(u - u_<\right)\left(\alpha - 2MGu\right) \tag{23.2}$$

where $\alpha = 1 - 2MG\left(u_< + u_>\right)$.

As the test body (planet) moves around its orbit, the (inverse) radius u will remain between the limits $u_<$ and $u_>$, expected to be close to their Newtonian values because $2MGu_>$ is very small. The angle φ—measured from $u_<$—will increase from 0 to whatever value it has when $u = u_>$.

The easiest way to calculate the change of angle uses the change of variable

$$u = \frac{1}{2}\left(u_> - u_<\right)\cos\chi + \frac{1}{2}\left(u_> + u_<\right) \tag{23.3}$$

leading to

$$\frac{du}{d\varphi} = -\frac{1}{2}\left(u_> - u_<\right)\sin\chi\,\frac{d\chi}{d\varphi} \tag{23.4}$$

$$= \pm\frac{1}{2}\left(u_> - u_<\right)\sin\chi\left[1 - 3MG\left(u_> + u_<\right) - MG\left(u_> - u_<\right)\cos\chi\right]^{1/2},$$

i.e.,

$$\frac{d\chi}{d\varphi} = \pm\left[1 - 3MG\left(u_> + u_<\right) - MG\left(u_> - u_<\right)\cos\chi\right]^{1/2}. \tag{23.5}$$

As the variable χ changes by 2π, the planet moves around once in its orbit. The Newtonian case corresponds to replacing the right side of 23.5 by unity (1), giving $\Delta\varphi = \Delta\chi = 2\pi$. (Of course $u_<$ and $u_>$ must be replaced with their Newtonian values also.)

However, the change in χ including the corrections of Einstein's theory of gravitation is

$$\Delta\varphi = \int_0^{2\pi} d\chi \left[1 - 3MG\left(u_> + u_<\right) - MG\left(u_> - u_<\right)\cos\chi\right]^{-1/2} \qquad (23.6)$$

$$\approx 2\pi\left[1 + \frac{3MG}{2}\left(u_> + u_<\right) + \ldots\right].$$

The amount, $\frac{6\pi MG}{R}$—where $\frac{1}{R} = \frac{1}{2}\left(\frac{1}{r_>} + \frac{1}{r_<}\right)$is the "semi latus rectum"—by which $\Delta\varphi$ differs from 2π, measures the precession of the orbit in space, per revolution. The difference is positive, so a planet precesses in the direction of its motion.

Generating approximations *via* contour integration

The student may legitimately ask, "How did we know to make the specific change of variable 23.3?" Of course, one answer is that this was fairly standard in orbital perturbation theory (a subject more widely taught in Einstein's time than our own), so this is essentially how Einstein tackled it. In other words, "Tradition!" But traditional methods, while they must be mastered by the aspiring theoretical physicist, offer no insight into new problems.

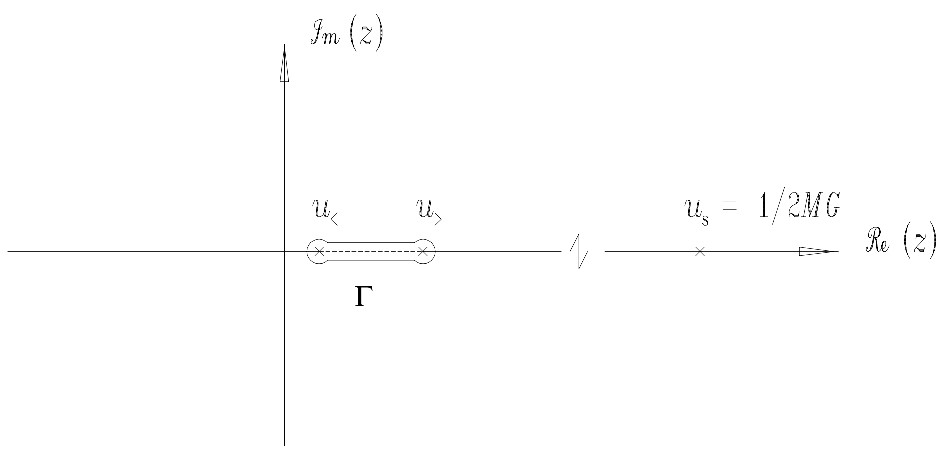
Therefore it seems worthwhiile to digress on another method for generating approximations in theoretical mechanics. The angular increase in an orbit can be written as an integral

$$\Delta\varphi = 2\int_{u_<}^{u_>} du\left[-u^2 + 2au - b + 2MGu^3\right]^{-1/2}. \qquad (23.7)$$

Eq. 23.7 can be re-expressed instructively as a contour integral,

$$\Delta\varphi = \oint_\Gamma dz\left[\left(u_> - z\right)\left(z - u_<\right)\left(\alpha - 2MGu\right)\right]^{-1/2}$$

where the closed contour Γ is as shown below:



We note the closed curve goes around (counter-clockwise) a dashed line drawn between the points $u_<$ and $u_>$. To understand why we have drawn the curve this way, and how we will use this construction to evaluate (approximately) the integral, we digress briefly on the art of contour integration.

Contour integration

The method of contour integration is based on Cauchy's theorem, which states that the integral around a closed contour Γ, of a function $f(z)$ that is analytic[†] within Γ and continuous on it is identically 0:

$$\oint_\Gamma f(z) = 0$$

Cauchy's theorem is proved in standard books on the theory of functions of a complex variable such as Whittaker and Watson, *A Course of Modern Analysis*, 4th ed. (Cambridge University Press, 1996). Its use inevaluating integrals is detailed in many places, such as Arfken and Weber, Mathews and Walker, *etc.*

A contour integral is defined as follows: suppose a closed curve in the complex plane can be parameterized by $z = \zeta(t)$, $0 \le t < 1$. Then

$$\oint_\Gamma f(z) = \int_0^1 dt \, \frac{d\zeta}{dt} \, f\big(\zeta(t)\big)$$

What does it mean to say a complex-valued function of a complex variable is analytic at a point z_0 in the complex plane? We mean that it can be differentiated in the usual manner:

$$\frac{df(z)}{dz} = \lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h}$$

where h is a complex number whose magnitude $|h|$ is allowed to become arbitrarily small.

Problem:

Show that if $f(x+iy) = a(x,y) + ib(x,y)$, then $\dfrac{df(z)}{dz}$ exists *iff*[‡]

$$\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x} = 0$$

(Cauchy-Riemann equations)

$$\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y} = 0$$

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† This term will be defined below.

‡ "…if and only if…"

Now we come to *singularities*. These are points where $f(z)$ is not analytic. They come in three flavors:

- *Poles*—near a simple pole a function behaves as $f(z) = \frac{R}{z - z_0} + g(z)$. Near a pole of order n the behavior is $f(z) = \frac{R}{(z - z_0)^n} + g(z)$ where n is a positive integer.
- *Essential singularities*— a function that has an infinite series of poles of orders 1, 2, ... is said to have an essential singularity at z_0 . An example is $e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$.
- *Branch cuts*—these are like a continuous line of poles. For example, the function

$$f(z) = \int_0^1 \frac{du}{u - z} = \log(1-z) - \log(-z)$$

has a branch cut from 0 to 1 along the real axis. If we define $f(z)$ to be real for negative real z, then obviously it is discontinuous across the real axis for $0 < \text{Re}(z) < 1$. Another example is $z^{1/2}$. Whenever a function has a discontinuity across a line, it is impossible to define the derivative in the direction perpendicular to that line. So we "cut" the line of discontinuity out of the plane (that is the meaning of the dashed line in the figure above) and say the function is analytic in the *cut plane*. To specify such a function we have to specify its branch cuts and the direction we cut them out. For example, it would have been equally permissible in the above logarithmic case to run the cuts from $-\infty \to 0$ and from $0 \to \infty$ along the real axis. That would define a different, but equally valid function. Generally we choose branch lines to suit our convenience.

Poles and essential singularities are *isolated*—one can draw a circle around them (of arbitarily small radius) and say that outside that circle the function is analytic because its derivative exists. Branch lines are like a continuous line of poles, hence are not isolated.

Integrals that can be evaluated by contour integration have certain things in common. First is that the integral is well-defined; second, that it be definite; and third, the range of integration can be made part of a suitable closed contour, in such a way that the integral along the rest of the contour either vanishes or can be performed explicitly. Virtually the entire art of contour integration rests on choosing suitable contours.

We now give two examples. First, consider the definite integral

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} .$$

We see that the polynomial in the denominator can be factored into the form

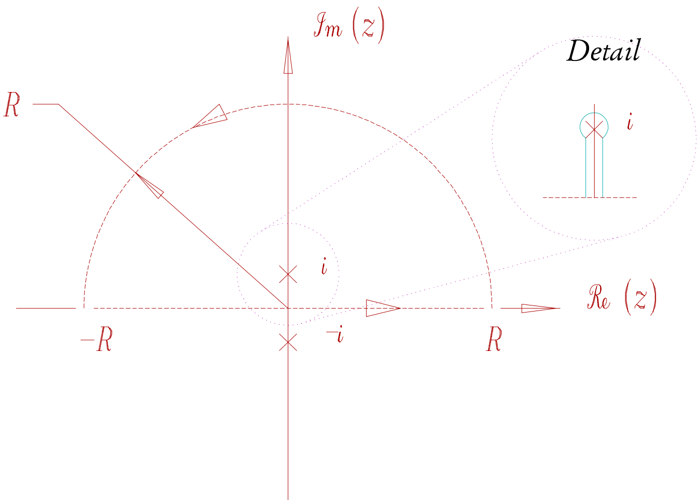
$$x^2 + 1 = (x - i)(x + i)$$

meaning the function has simple poles at $x = \pm i$ in the complex plane.

In this case we want a contour that includes the real axis, from $-R$ to $+R$, plus some part that closes the contour and on which the integral can be easily evaluated. In this case, the choice (dictated by experience) is a semi-circle in the upper half of the complex plane, as shown below.

The integral we want can be written

$$I = \lim_{R \to \infty} \int_{-R}^{+R} \frac{dx}{1 + x^2}$$



The portion on the semicircle can be represented by letting

$$z = Re^{i\theta}$$

giving

$$I_{semi} = \lim_{R \to \infty} \int_0^{\pi} \frac{i\,\mathrm{R}e^{i\theta}\,d\theta}{1 + R^2\,e^{2i\theta}}$$

$$\to \lim_{R \to \infty} \frac{1}{R} \int_0^{\pi} i\,e^{-i\theta}\,d\theta = 0\,.$$

That is, we can add the contribution from the large semicircle because in the limit of large radius, it goes to 0.

Now the integral is in the form of a contour integral, but the integrand is not analytic everywhere within the contour. To put it in the form of such an integral we draw a vertical line from the point $x + iy = -\varepsilon + i0$ to the point $-\varepsilon + i(1 - \varepsilon)$, a similar line from $\varepsilon + i(1 - \varepsilon)$ to $\varepsilon + i0$, and a circle of radius ε centered at $z = 0 + i$, going clockwise about the pole at i and connecting the two lines. These are shown in the Detail. The total contour integral can now be written

$$\lim_{\substack{r \to \infty \\ \varepsilon \to 0}} \left(\int_{-R}^{-\varepsilon} + \int_{+\varepsilon}^{+R} \right) + I_{semi} + \lim_{\varepsilon \to 0} \int_{2\pi}^{0} \frac{\varepsilon\,e^{i\theta}\,d\theta}{1 - (1 - \varepsilon e^{i\theta})^2} = 0$$

where we have parameterized z on the little circle as

$$z = i - i\varepsilon\,e^{i\theta}\,.$$

The two contributions from the vertical lines cancel, since the same function is being integrated alone the same contour in opposite directions, so they have not been displayed explicitly.

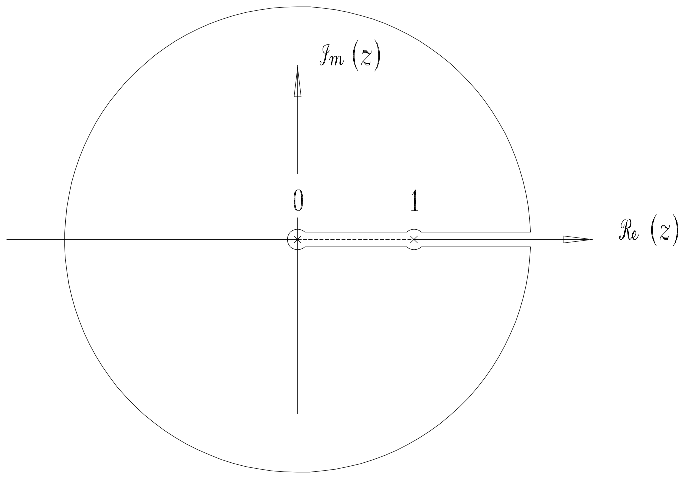
The first term clearly becomes the integral we want, the second vanishes in the limit of large R, and the last term becomes

$$\lim_{\varepsilon \to 0} \int_{2\pi}^{0} \frac{\varepsilon\,e^{i\theta}\,d\theta}{1 - 1 + 2\varepsilon e^{i\theta} - \varepsilon^2 e^{2i\theta}} = \lim_{\varepsilon \to 0} \int_{2\pi}^{0} \frac{d\theta}{2 - \varepsilon e^{i\theta}} = -\pi\,.$$

We are left with $I - \pi = 0$, or $I = \pi$ (which is well-known to be correct—as can be seen from the trigonometric substitution $x \to \tan\theta$).

How do we do an integral with a branch cut? Consider $I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

The function $f(z) = \left(z(1-z)\right)^{-1/2}$ may be defined with a branch cut running from 0 to 1 along the real axis. If we define the function to be real and equal to $\dfrac{1}{\sqrt{x(1-x)}}$ as z approaches the cut from above, then the integral on a contour such as shown below vanishes. The pieces on the extension lines running above and below the real axis from 1 to ∞ cancel because the function, as defined, has no discontinuity across the real axis in that range. The contributions from the tiny circular end caps vanish as $\sqrt{\varepsilon}$, as they must for the integral to have a meaning at the endpoints (where the integrand blows up). The function as defined has the opposite sign when z approaches the cut from below, hence the contour integral (whose total value is 0, by Cauchy's Theorem) becomes



$$2I + \lim_{R \to \infty} \int_0^{2\pi} i\, R\, e^{i\theta}\, d\theta \left(R\, e^{i\theta}\,(1 - R\, e^{i\theta})\right)^{-1/2} = 0\,.$$

Now clearly, as we allow R to get larger than 1, the square root of the – sign beomes either $-i$ or $+i$. The question is, which is it? Putting aside this question for a moment, we see that for large R the integrand of the integral on the large circle becomes ±1 (plus something that vanishes as $r \to \infty$), so the value of this integral becomes $\pm 2\pi$. In other words the integral we want will have the value $\pm\pi$. Obviously, since the integrand is positive, the + sign must be taken. Is there some way we could have known this *a priori*?

Consider the imaginary part of the function $\left[(x + i\varepsilon)\,(1 - x - i\varepsilon)\right]^{-1/2}$ where ε is very small—as long as x lies between 0 and 1, this imaginary part is proportional to $\varepsilon\,(2x-1)$, hence it is positive for $x > 1/2$. By continuity we see that if $\pi/2 > \theta > 0$ the imaginary part of $\left[R\, e^{i\theta}\,(1 - R\, e^{i\theta})\right]^{-1/2}$ must be positive for R near 1, *i.e.* we want a factor of $+i$.

Application to perihelion precession

We now apply this technique to the approximate evaluation of the integral

$$\Delta\varphi = \oint_\Gamma dz \left[(u_> - z)(z - u_<)(\alpha - 2MGu)\right]^{-1/2}$$

We extend the contour Γ to include a large circle centered at $u_<$ and integrate around Γ. Cauchy's theorem assures us the answer is 0. Let us parameterize the circle as

$$z = u_< + \rho\, e^{i\theta}$$

where ρ is much larger than $u_> - u_<$, but much smaller than $u_S = \alpha/2MG$. Then for any ρ

$$\Delta\varphi + \int_0^{2\pi} i\rho\, e^{i\theta}\, d\theta\left[\left(u_> - u_< - \rho e^{i\theta}\right)\rho e^{i\theta}\left(\alpha - 2MGu_< - 2MG\rho e^{i\theta}\right)\right]^{-1/2} = 0\ .$$

If we factor out the – sign as before, we may rewrite this as

$$\Delta\varphi = \int_0^{2\pi} d\theta\left[\left(1 - \frac{(u_> - u_<)\, e^{-i\theta}}{\rho}\right)\left(1 - 2MG(u_> + 2u_<) - 2MG\rho e^{i\theta}\right)\right]^{-1/2}$$

$$\approx \int_0^{2\pi} d\theta\left(1 + \frac{(u_> - u_<)\, e^{-i\theta}}{2\rho} + \ldots\right)\left(1 + MG(u_> + 2u_<) + MG\rho e^{i\theta} + \ldots\right)$$

It is easy to see that because

$$\int_0^{2\pi} d\theta\, e^{\pm i n \theta} = 0$$

for any non-zero integer n, the result is an expansion in powers of MGu, of which the leading term is

$$\Delta\varphi - 2\pi = 3\pi MG\left(u_> + u_<\right),$$

as before, and which is independent of ρ.