# Neutron stars and white dwarfs

A neutron star or a white dwarf star is so hot that to first approximation we may regard all its atoms as fully ionized. (Of course, most of the matter in a neutron star is neutrons, which do not ionize.)

The baryonic matter in a white dwarf is mainly $^4$He. This provides the mass, while the electrons provide charge neutrality and supply the pressure that keeps the star from collapsing under its own weight.

Assume $Z$ electrons. Clearly, if $N$ is the number of $^4$He nuclei, $Z = 2N$.

The mass of the star is then $N m_{He} \approx 4 N m_{proton} \equiv 4Nm$ .

The electrons are fermions, so in a box of volume $\Omega$ large enough to contain many electrons, but small enough that the tidal effects of the gravitational field are small across its volume, the electrons fill up all the available levels to the Fermi energy.

An electron state in the box is labelled by momentum $\vec{p}$ and spin $\sigma$, which can be "up" or "down". Hence the number of electron states with momenta between $\vec{p}$ and $\vec{p} + d\vec{p}$ is the Fermi distribution,

$$dZ = \frac{2\Omega\, d^3 p}{h^3} \frac{1}{e^{(\varepsilon(p) - \varepsilon_F)/kT} + 1} \quad . \tag{24.1}$$

The temperature $kT$ is relatively low compared with the Fermi energy $\varepsilon \approx 0.13$ MeV, so the Fermi distribution becomes a $\theta$-function. That is, we have

$$Z = \frac{2\Omega}{(2\pi\hbar)^3} \frac{4\pi p_F^3}{3} \tag{24.2}$$

leading to a local electron density

$$n_e = \frac{Z}{\Omega} = \frac{p_F^3}{3\pi^2 \hbar^3} \quad . \tag{24.3}$$

Thus the local average kinetic energy of the electrons is

$$\langle \varepsilon \rangle = \frac{1}{Z} \int dZ(p)\ \varepsilon(p) \tag{24.4}$$

and the local (kinetic) energy density is the average kinetic energy of an electron times the number density. For nonrelativistic electrons,

$$\langle \varepsilon \rangle = \frac{3}{5} \frac{p_F^2}{2m_e} \tag{24.5}$$

whereas for ultra-relativistic ones,

$$\langle \varepsilon \rangle \approx \frac{3}{4} p_F\, c \quad . \tag{24.6}$$

We see that as the star gets denser, the electrons can become ultra-relativistic.

**Problem:**
A white dwarf star has the solar mass and a radius $\approx 9000\ \text{Km}$. What is the average Fermi momentum?

Are the electrons relativistic or nonrelativistic?

What is the temperature corresponding to this Fermi momentum?

Thus the kinetic energy density for nonrelativistic electrons is

$$U(r) = n_e \langle \varepsilon \rangle = \frac{3}{5} \frac{p_F^2}{2m_e} \frac{p_F^3}{3\pi^2 \hbar^3} \tag{24.7}$$

where we imagine $p_F$ is a function of position, as the local density varies.

Since $n_e = \frac{Z}{\Omega} = \frac{2\rho}{4m} = \frac{\rho}{2m}$ ($\rho$ is the mass-density) we can re-express Eq. 24.7 as

$$U(r) = \frac{3\hbar^2}{10m_e}(3\pi^2)^{2/3}\left(\frac{\rho}{2m}\right)^{5/3}. \tag{24.8}$$

The total mass within a sphere of radius r is

$$M(r) = 4\pi \int_0^r dr' \, r'^2 \, \rho(r')$$

and the gravitational acceleration at $r$ is therefore

$$a(r) = -\frac{M(r)\,G}{r^2}. \tag{24.9}$$

The pressure at $r$ is obtained from the (isentropic) equation for work:

$$d\left(\frac{U}{\rho}\right) + p\,d\Omega = \frac{dU}{\rho} - U\frac{d\rho}{\rho^2} - p\frac{d\rho}{\rho^2} = 0 \tag{24.10}$$

or

$$p = \rho\frac{dU}{d\rho} - U = \frac{2}{3}U \stackrel{df}{=} \Gamma\rho^{5/3}. \tag{24.11}$$

The equation for static equilibrium states that the gravitational force on a small volume of area $A$ and height $dr$ should equal the difference of the pressure between $r$ and $r + dr$ :

$$\frac{1}{r^2}GM(r)\,\rho(r)\,A\,dr = A\big(p(r) - p(r+dr)\big) \tag{24.12}$$

or

$$GM(r)\,\rho(r) = -r^2\frac{dp}{dr}$$

$$4\pi G\int_0^r dx\,x^2\,\rho(x) = -\frac{5}{3}\Gamma\,r^2\,\rho^{-1/3}\frac{d\rho}{dr} \tag{24.13}$$

which can finally be simplified to Chandrasekhar's equation

$$4\pi G\, r^2\, \rho(r) \;=\; -\,\tfrac{5}{3}\,\Gamma\, \frac{d}{dr}\left(r^2\, \rho^{-1/3}\, \frac{d\rho}{dr}\right). \qquad (24.14)$$

Equation 24.14 must be solved by numerical methods. Thus we shall try to get an approximate result. Go back to Eq 24.7 and estimate the total kinetic and potential energy of the star:

$$KE \;\approx\; Z\,\frac{3}{5}\,\frac{p_F^2}{2m_e} \;\approx\; \frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3} \frac{Z^{5/3}}{2m_e\, R^2} \qquad (24.15)$$

$$V \;\approx\; -\,\frac{3}{5}\,\frac{M_{star}^2\, G}{R_{star}} \;=\; -\,\frac{3}{5}\,\frac{(2ZM)^2\, G}{R} \qquad (24.16)$$

We see that the total energy has the form

$$E_{tot} \;=\; \frac{a}{R^2} \;-\; \frac{b}{R} \qquad (24.17)$$

which has a minimum when $R \;=\; 2a/b$, whereupon

$$E_{\min} \;=\; \frac{-b^2}{4a} \qquad (24.18)$$

## Ultra-relativistic electrons

What happens when the electrons get so dense they are relativistic? From Eq. 24.6 we see that Eq. 24.17 becomes

$$E_{tot} \;=\; \frac{a'}{R} \;-\; \frac{b}{R} \qquad (24.19)$$

which has no minimum if $a' < \; b$, and no maximum of the sign of the inequality is reversed. More specifically,

$$\begin{aligned} KE \;&=\; Z\,\frac{3}{k_F^3}\int_0^{k_F} dk\, k^2\, \left(\sqrt{k^2 + m_e^2} \;-\; m_e\right) \;\approx\; \frac{3}{4}\, k_F\, Z \\ V \;&\approx\; -\,\frac{3}{5}\,\frac{4M_N^2\, Z^2}{R} \end{aligned} \qquad (24.20)$$

where $k_F$ is now the average electron Fermi momentum of the star, given by

$$k_F \;=\; \left(\frac{3\pi^2 \cdot 3Z}{4\pi\, R^3}\right)^{1/3} \;=\; \frac{1}{R}\,\left(\frac{9\pi Z}{4}\right)^{1/3} \;.$$

Hence we have

$$\begin{aligned} a' \;&=\; \frac{3}{4}\, Z^{4/3} \left(\frac{9\pi}{4}\right)^{1/3} \\ b \;&=\; \frac{3}{5}\, Z^2\, 4M^2\, G \;. \end{aligned} \qquad (24.21)$$

Since $Z^2$ beats $Z^{4/3}$ as $Z \to \;\infty$, eventually a sufficiently massive white dwarf star will become unstable against further collapse. That is, the pressure of the degenerate electron gas is not enough---when the electrons become relativistic----to support the star against its own weight. This takes place (roughly) when $Z = Z_{crit}$ , where

$$Z_{crit}^{2/3} \approx \frac{5}{4}\left(\frac{9\pi}{4}\right)^{1/3} \frac{\hbar c}{4M^2\, G} \quad . \tag{24.22}$$

Note that $\left[\frac{\hbar\, c}{G}\right] = [M^2]$; this suggests we define the *Planck mass*

$$M_{Planck} = \left(\frac{\hbar\, c}{G}\right)^{1/2} \tag{24.23}$$

In terms of $M_{Planck}$ we find

$$Z_{crit} = \left(\frac{5}{16}\right)^{3/2}\left(\frac{9\pi}{4}\right)^{1/2}\left(\frac{M_{Pl}}{M_N}\right)^3 \quad . \tag{24.24}$$

The stellar mass corresponding to this (approximate) Chandrasekhar limit is

$$M_{crit} = 2M_N\, Z_{crit} \approx 0.464 \times 2 \times \frac{M_{Pl}^3}{M_N^2} = 3.16 \times 10^{33}\ \text{gm} = 1.6\, M_{\odot} \tag{24.25}$$

When we perform the calculation more accurately, *i.e.* when we take into account the radial variation of density and the corrections to the extreme relativistic kinematics of the electrons, we find instability setting in at a lower value of $M_{star}$ , about 1.45 $M_{\odot}$ .

## Neutron stars

What happens when $M_{star} > M_{crit}$ ? Clearly, the star collapses as it cools, the electrons become ultra-relativistic, and the collapse continues until something else can halt it.

Normally, a free neutron is unstable with respect to $\beta$-decay:

$$n \to p + e^- + \bar{\nu}_e + 0.8\ \text{MeV}.$$

If the neutron is surrounded by an electron gas, whose pressure increases to the point where the maximum kinetic energy of the emitted electron ($\varepsilon_e \approx 0.8$ MeV) is below the Fermi energy, the neutron can no longer $\beta$-decay because there are no available states for the electron. We call this effect “Pauli blocking”.

Moreover, when the electron pressure is this large, the inverse $\beta$-decay reaction,

$$p + e^- \to n + \nu_e - 0.8\ \text{MeV},$$

beomes possible. The inverse process takes place when

$$\sqrt{m_e^2 + k_F^2} - m_e = 0.8\ \text{MeV} \tag{24.26}$$

or $k_F \approx 1.2$ MeV/c. Then with $Z = Z_{\odot} \approx 10^{57}$,

$$n_e = \frac{k_F^3}{3\pi^2} = 7.5\times10^{30}\ /\text{cm}^3 = \frac{3Z}{4\pi R^3} \qquad (24.27)$$

which gives

$$R \approx 3\times10^8 \text{ cm} = 3000 \text{ km} .$$

In other words, inverse β-decay becomes favorable at about the same time the white-dwarf star becomes unstable for collapse. As the collapse proceeds, all the protons in the interior of the star inverse β-decay to neutrons which cannot β-decay because there is an atmosphere of protons and electrons, with enough electron density to Pauli-block the β-decay.

In other words, a collapsed white dwarf turns into a Fermi gas consisting of $N \approx 2Z$ neutrons, with total kinetic and potential energies

$$KE = \begin{cases} \frac{3}{5} N \frac{n_F^2}{2M_N}, & \text{(nonrelativistic)} \\ \\ \frac{3}{4} N\, n_F, & \text{(relativistic)} \end{cases} \qquad (24.28a)$$

$$V = -\frac{3}{5}\frac{N^2 M_N^2 G}{R} + V_{Nuc} \qquad (24.28b)$$

where

$$n_F = \left(\frac{9\pi N}{4}\right)^{1/3} \frac{1}{R} .$$

The nuclear interaction energy $V_{Nuc}$ becomes significant when the (number) density becomes comparable to nuclear densities, $n_{Nuc} = 1.6\times10^{38}\ /\text{cm}^3$ . Then

$$n_F \approx 250 \text{ MeV/c} ,$$

$$R \approx 2\times10^6 \text{ cm} = 20 \text{ Km} .$$

## Role of General relativity

The dimensionless gravitational potential $\Phi$ is $\frac{MG}{Rc^2}$ where $M = M_\odot$ . The Schwarzschild metric involves 2Φ, which for white dwarf stars at the Chandrasekhar limit is $10^{-3}$; whereas for neutron stars $2\Phi \approx 0.15$ . Hence general relativity is important for neutron stars, but not for white dwarves.

**Gravitation and Cosmology**

Role of General relativity