General relativistic theory of stellar equilibrium

The gravitational equations

The Einstein equations for a distributed (fluid) source are, as before,

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \tag{25.1}$$

where we take as the energy-momentum tensor of a relativistic fluid

$$T_{\mu\nu} = -p\, g_{\mu\nu} + \left(\rho + p\right) U_\mu U_\nu . \tag{25.2}$$

The 4-velocity U_μ satisfies

$$g^{\mu\nu} U_\mu U_\nu = 1 , \tag{25.3}$$

hence

$$T \overset{df}{=} g^{\mu\nu} T_{\mu\nu} = -4p + \rho + p = \rho - 3p \tag{25.4}$$

and

$$R_{\mu\nu} = -8\pi G\left(\frac{1}{2} g_{\mu\nu} (p - \rho) + (p + \rho) U_\mu U_\nu\right). \tag{25.5}$$

We assume a metric of Schwarzschild form

$$g_{tt} = B(r) , \qquad g_{rr} = -A(r)$$

$$g_{\theta\theta} = -r^2 , \qquad g_{\varphi\varphi} = -r^2 \sin^2\theta$$

take the fluid to be static, $U_r = U_\theta = U_\varphi = 0$, and from Eq. 25.3, find

$$U_t = \sqrt{B(r)} . \tag{25.6}$$

Now we work out the components of the curvature tensor, as before:

$$R_{tt} = \frac{-B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} = -4\pi G\,(3p + \rho)\, B \tag{25.7t}$$

$$R_{rr} = \frac{-B''}{2B} + \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{A'}{A} = 4\pi G\,(\rho - p)\, A \tag{25.7r}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} = -4\pi G\,(\rho - p)\, r^2 \tag{25.7θ}$$

$$R_{\varphi\varphi} = \sin^2\theta \; R_{\theta\theta} \tag{25.7φ}$$

Eq. 25.7θ, φ are identical, and testify to the rotational invariance of the problem.

We now want the equation of hydrostatic equilibrium. Recall from §24 that in the Newtonian case we had to balance the force of gravitation against the pressure gradient, at a distance *r* from the origin:

$$-r^2 \frac{dp}{dr} = G\, M(r)\, \rho(r)\,. \qquad (24.13)$$

However, in the general-relativistic case there is no such thing as a "force of gravitation". Where, then, does gravitation enter the condition of hydrostatic equilibrium? We see that it must come from the covariant generalization of the conservation of energy. (We could also have derived Newtonian hydrostatic equilibrium from an energy principle.) That is,

$$T^{\mu\nu}{}_{;\,\nu} = 0 \qquad (25.8)$$

or

$$T^{\mu\nu}{}_{,\,\nu} + \begin{Bmatrix}\mu \\ \sigma\,\nu\end{Bmatrix} T^{\sigma\nu} + \begin{Bmatrix}\nu \\ \sigma\,\nu\end{Bmatrix} T^{\mu\sigma} = \begin{Bmatrix}\mu \\ \sigma\,\nu\end{Bmatrix} T^{\sigma\nu} + \frac{1}{\sqrt{g}}\partial_\nu\left(\sqrt{g}\, T^{\mu\nu}\right) = 0\,. \qquad (25.9)$$

Since

$$\left(-p\, g^{\mu\nu}\right)_{;\,\nu} \equiv -g^{\mu\nu}\, p_{,\,\nu} \qquad (25.10)$$

and since only $U_t \neq 0$ (also it is independent of time) we have

$$-g^{\mu\nu}\, p_{,\,\nu} + \begin{Bmatrix}\mu \\ t\,t\end{Bmatrix} U^t\, U^t\,(p + \rho) \equiv -g^{\mu\nu}\, p_{,\,\nu} + \begin{Bmatrix}\mu \\ t\,t\end{Bmatrix} \frac{1}{B(r)}\,(p + \rho) = 0\,. \qquad (25.11)$$

Now, since the only variation is with respect to r (by symmetry), all derivatives except with respect to r vanish, leading to

$$\begin{Bmatrix}\mu \\ t\,t\end{Bmatrix} = -\frac{1}{2} g^{\mu\nu}\, B_{,\,\nu} = -\frac{1}{2} g^{rr}\, B_{,\,r}$$

thence to

$$-g^{rr}\, p_{,\,r} - \frac{1}{2} g^{rr}\, B_{,\,r}\, \frac{1}{B}\,(p+\rho) = 0\,,$$

and finally to

$$\frac{dp}{dr} + \frac{B'}{2B}\,(p+\rho) = 0\,. \qquad (25.12)$$

Now, to solve the equations for the metric tensor, take the linear combination

$$\frac{R_{tt}}{2B} - \frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} = -\frac{1}{r^2} - \frac{A'}{2rA^2} + \frac{1}{r^2\, A} = -8\pi G\rho \qquad (25.13)$$

or

$$\frac{d}{dr}\left(\frac{r}{A}\right) = 1 - 8\pi G\rho(r)\, r^2$$

i.e.

$$A(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1} \qquad (25.14)$$

where as before

$$M(r) = 4\pi \int_0^r dx\, x^2\, \rho(x)\quad . \qquad (25.15)$$

We now use the equation Eq. 25.7θ together with Eq. 25.12 to relate the pressure gradient and density:

$$-1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} = -4\pi G\,(\rho - p)\,r^2\,;$$

insert $A(r)$ from Eq. 25.14 to get

$$\frac{B'}{2B} = AG\left(4\pi p(r)\,r + \frac{M(r)}{r^2}\right)$$

and $\frac{B'}{2B}$ from Eq. 25.12 to find at last

$$-r^2\,\frac{dp}{dr} = GM(r)\,\rho(r)\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{4\pi r^2\,p(r)}{M(r)}\right)\left(1 - \frac{2GM(r)}{r}\right)^{-1} \qquad (25.16)$$

To proceed further we require an equation of state, namely a relation between $p(r)$ and $\rho(r)$. We might, for example, model a neutron star as a (locally) non-interacting Fermi gas of neutrons, in which case

$$\rho(r) = \frac{1}{\pi^2}\int_0^{k_F(r)} dk\,k^2\left(k^2 + M_N^2\right)^{1/2} \qquad (25.17\text{ a})$$

$$p(r) = \frac{1}{3\pi^2}\int_0^{k_F(r)} dk\,k^4\left(k^2 + M_N^2\right)^{-1/2} \qquad (25.17\text{ b})$$

The relation between $p(r)$ and $\rho(r)$ is in principle established by solving, say, Eq. 25.17a for the local Fermi momentum $k_F(r)$—which can then be expressed in terms of $\rho(r)$—then inserting that expression in Eq. 25.17b.

It is found that under the pure Fermi gas assumption, a neutron star with mass $\approx M_\odot$ and radius $\approx$ 10 Km is the largest possible. However, with these parameters the central density exceeds the density of nuclei, hence the nuclear forces must play an important role. One may safely disregard the limits in Weinberg's book, since the theory of nuclear forces that led to it is now considered obsolete. At present we do not know the maximum possible mass of a neutron star.

What is it that determines M_{max} ? It turns out that if the mass gets too large, then no equation of state can produce enough pressure to sustain the weight. The reason for this is that the right hand side of Eq. 25.16 is made larger by the pressure and by the effects of gravitational distortion of space-time. It is fairly easy to see that if M is too large, as we integrate inward from some radius where $p=0$, we will reach a singularity in p—that is, the pressure can become infinite—while r is still >0. There is no equation of state that can supply infinite pressure at finite density, hence there is always a maximum M.