Dimensional analysis of neutron stars

The equations for the Ricci tensor were

$$R_{tt} = \frac{-B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} = -4\pi G\,(3p+\rho)\,B \tag{25.7t}$$

$$R_{rr} = \frac{-B''}{2B} + \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{A'}{A} = 4\pi G\,(\rho - p)\,A \tag{25.7r}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} = -4\pi G\,(\rho - p)\,r^2 \tag{25.7θ}$$

$$R_{\varphi\varphi} = \sin^2\theta\; R_{\theta\theta} \tag{25.7φ}$$

Size and mass

We shall now derive some qualitative results using dimensional analysis. Recall

$$-r^2\frac{dp}{dr} = GM(r)\,\rho(r)\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{4\pi r^2\,p(r)}{M(r)}\right)\left(1 - \frac{2GM(r)}{r}\right)^{-1} \tag{25.16}$$

and take for the equation of state of a pure neutron gas

$$\rho(r) = \frac{1}{\pi^2}\int_0^{k_F(r)} dk\,k^2\left(k^2 + m^2\right)^{1/2} \tag{25.17a}$$

$$p(r) = \frac{1}{3\pi^2}\int_0^{k_F(r)} dk\,k^4\left(k^2 + m^2\right)^{-1/2} \tag{25.17b}$$

We can rewrite Eq. 25.17b as

$$p(r) = \tfrac{1}{3}\,\rho(r) - \frac{m^2}{3\pi^2}\int_0^{k_F(r)} dk\,k^2\left(k^2 + m^2\right)^{-1/2} \tag{26.1}$$

Now change to the dimensionless variable $\sinh\theta = \dfrac{k}{m}$; let

$$\rho_c = \frac{m^4}{3\pi^2}\,, \quad \theta_c = \sinh^{-1}(k/m) \tag{26.2}$$

then

$$\rho(r) = 3\rho_c\int_0^{\theta_c} d\theta\,\cosh^2\theta\,\sinh^2\theta \tag{26.3a}$$

$$p(r) = \rho_c\int_0^{\theta_c} d\theta\,\sinh^4\theta\,. \tag{26.3b}$$

From Eq. 26.3a,b we have

$$p(r) = \rho_c F\left(\frac{\rho(r)}{\rho_c}\right)$$

where $F(x)$ is some transcendental function.

The dimensional quantities in the theory are therefore ρ_c and $2G$. From them we can construct a radius (recall $\hbar = c = 1$)

$$R_c = (2G\,\rho_c)^{-1/2} \tag{26.4}$$

and a mass

$$M_c = \frac{R_c}{2G} \approx 3.5\,M_{\odot}\,. \tag{26.5}$$

In general, the mass of the star must be M_c times a function of the dimensionless ratio $\frac{\rho(0)}{\rho_c}$:

$$M = M_c\, f\left(\frac{\rho(0)}{\rho_c}\right)$$

and the radius of the star must be R_c times a dimensionless function:

$$R = R_c\, g\left(\frac{\rho(0)}{\rho_c}\right).$$

Rotational frequency

Dimensional analysis also gives us a handle on the rotational frequencies of neutron stars: clearly the maximum frequency occurs when the centrifugal acceleration and gravitational acceleration are comparable, $\frac{GM}{R^2} \approx R\omega^2$:

$$\omega_{\max} \approx \left(\frac{GM}{R^3}\right)^{1/2} \approx \frac{1}{\sqrt{2}}\frac{c}{R_c} \approx 2\times10^4\ \mathrm{sec}^{-1} = \frac{2\pi}{\tau_{\min}}$$

or

$$\tau_{\min} \approx 0.3\times10^{-3}\ \mathrm{sec}\ ;$$

so that the observed pulsars with millisecond periods agree well with this. White dwarf periods, however, are necessarily much longer. Therefore pulsars must be neutron stars.

Some features of stellar structure

The total number of neutrons is

$$N = \int d^3r \left(A(r)\,B(r)\right)^{1/2} J_N^0(r) \tag{26.6}$$

where the (conserved) neutron current is $J_N^\mu(r)$. We can define the "proper" number density as

$$n(r) \stackrel{df}{=} U_\mu J_N^\mu = \left(B(r)\right)^{1/2} J_N^0(r) \tag{26.7}$$

hence

$$N = 4\pi \int dr\, r^2 \left(A(r)\right)^{1/2} n(r)\,. \tag{26.8}$$

We can define the energy content of the star as

$$E_{tot} = T + V = M(R) - Nm \tag{26.9}$$

and the local (non-gravitational) energy density as

$$\varepsilon(r) = \rho(r) - m\,n(r) \tag{26.10}$$

which gives

$$\begin{aligned}
E &= 4\pi \int dr\, r^2 \left[\rho(r) - m\left(A(r)\right)^{1/2} n(r)\right] \\
&= 4\pi \int dr\, r^2 \left[\varepsilon(r) + m\,n(r)\,(1 - \left(A(r)\right)^{1/2})\right] \\
&= 4\pi \int dr\, r^2 \left[\varepsilon(r) + \left(\rho(r) - \varepsilon(r)\right)(1 - \left(A(r)\right)^{1/2})\right] \\
&= 4\pi \int dr\, r^2 \left[\varepsilon(r)\left(A(r)\right)^{1/2} + \rho(r)\,(1 - \left(A(r)\right)^{1/2})\right].
\end{aligned} \tag{26.11}$$

We can make the identifications

$$T = 4\pi \int dr\, r^2\, \varepsilon(r)\left(A(r)\right)^{1/2} \tag{26.12a}$$

$$V = 4\pi \int dr\, r^2\, \rho(r)\left[1 - \left(A(r)\right)^{1/2}\right] \tag{26.12 b}$$

and to leading order we see that

$$T \approx 4\pi \int dr\, r^2\, \varepsilon(r) \tag{26.13 a}$$

and since $A(r) \approx 1 + \frac{MG}{r}$,

$$V = -4\pi \int dr\, r^2\, \rho(r) \frac{GM(r)}{r} \equiv -4\pi \int dM(r) \frac{GM(r)}{r} \tag{26.13 b}$$

Equations 26.13a,b are precisely what we would have written down based on Newtonian mechanics and Newtonian gravitation.