Equations of motion in general relativity

Reading: Adler, Bazin & Schiffer, *Introduction to General Relativity*, Chapter 10.

How many equations?

We recall that in the theory of stellar structure, the equations for the gravitational field were

$$R_{tt} = \frac{-B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\frac{B'}{A} = -4\pi G\,(3p+\rho)\,B \tag{25.7t}$$

$$R_{rr} = \frac{-B''}{2B} + \frac{B'}{4B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{A'}{A} = 4\pi G\,(\rho-p)\,A \tag{25.7r}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) + \frac{1}{A} = -4\pi G\,(\rho-p)\,r^2 \tag{25.7θ}$$

$$R_{\varphi\varphi} = \sin^2\theta\ R_{\theta\theta} \tag{25.7φ}$$

We also had an equation of hydrodynamic equilibrium

$$\frac{dp}{dr} + \frac{B'}{2B}(p+\rho) = 0\,. \tag{25.12}$$

and an equation of state

$$p(r) = \rho_c\,F\left(\frac{\rho(r)}{\rho_c}\right).$$

Now, Eq. 25.7φ is redundant with Eq. 25.7θ because of rotational invariance. But it might seem as though we had 4 unknowns ($A(r)$, $B(r)$, $\rho(r)$ and $p(r)$) and 5 equations to determine them. Are these unknowns overdetermined? Is it possible to solve them at all?

Clearly, the equation of hydrodynamic equilibrium is an identity that follows from

$$\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right)_{;\,\nu} = T^{\mu\nu}{}_{;\,\nu} = 0 \tag{27.1}$$

and so is not independent from it.

Einstein was the first to remark on the rather striking fact that the equations of motion describing the motion of particles under gravitational forces follow from the field equations---rather than being distinct from them. The subject was first elaborated by Einstein, Infeld and Hofmann† and then simplified and extended by Einstein and Infeld‡. In this lecture I can only give you some of the flavor of the theory, rather than the whole subject.

† Einstein, Infeld and Hofmann, *Ann. of Math.* **39** (1938) 66.

Motion of particles

The basic equation is $T^{\mu\nu}{}_{;\nu} = 0$.

Isolate a piece of matter in a world-tube D: Clearly,

$$\sqrt{g}\, T^{\mu\nu}{}_{;\nu} = \partial_\nu (\sqrt{g}\, T^{\mu\nu}) + \sqrt{g} \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} T^{\sigma\nu} = 0 \quad (27.2)$$

Thus,

$$\int_D d^4x\, \partial_\nu (\sqrt{g}\, T^{\mu\nu}) + \int_D d^4x \sqrt{g} \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} T^{\sigma\nu} = 0 \quad (27.3)$$

Now, since the matter is inside D, $T^{\mu\nu}$ is zero on S_2, hence

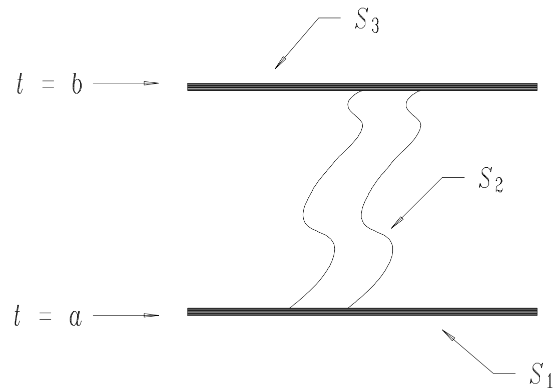

Diagram of the world-tube D

$$\int_D d^4x\, \partial_\nu (\sqrt{g}\, T^{\mu\nu}) = \int_{S_1+S_3} d^4x\, \partial_\nu (\sqrt{g}\, T^{\mu\nu}) = \int_{S_1+S_3} d^3x \sqrt{g}\, T^{\mu\nu} n_\nu \quad (27.4)$$

where on S_1 the unit normal is $n_\nu = \left(-1, 0, 0, 0\right)$ and on S_3 it is $n_\nu = (1, 0, 0, 0)$. Now, suppose we employ locally freely-falling coordinates to describe the barycenter of the particles, for which

$$\sqrt{g}\, \frac{dt}{d\tau} = 1 . \quad (27.5)$$

Then if we ignore pressure (that is, $T^{\mu\nu}$ refers to isolated, non-interacting particles----dust)

$$T^{\mu\nu} = \rho\, U^\mu U^\nu \quad (27.6)$$

and

$$\int_{S_1+S_3} d^3x \sqrt{g}\, T^{\mu\nu} n_\nu = \int d^3x\, \rho\, U^\mu \left.\frac{dt}{d\tau}\right|_{t=b} - \int d^3x\, \rho\, U^\mu \left.\frac{dt}{d\tau}\right|_{t=a}$$

$$\equiv \int_{t=a}^{t=b} d\tau\, m \frac{d^2x^\mu}{d\tau^2} \quad (27.7)$$

where $m \overset{df}{=} \int d^3x\, \rho$.

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‡ Einstein and Infeld, *Can. J. Math.* **1** (1949) 209.

Similarly we use Eq. 27.5 to write

$$\int_D d^4x \sqrt{g} \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} T^{\sigma\nu} = \int_D d^3x\, d\tau \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} \rho\, U^\sigma\, U^\nu \approx \int_{t=a}^{t=b} d\tau \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} m \frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} \qquad (27.8)$$

so that

$$\int_{t=a}^{t=b} d\tau\; m \left[\frac{d^2x^\mu}{d\tau^2} + \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} \frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} \right] = 0\,. \qquad (27.9)$$

That is, we recover the equation of motion for a test particle in a gravitational field,

$$\frac{d^2x^\mu}{d\tau^2} + \begin{Bmatrix} \mu \\ \sigma\,\nu \end{Bmatrix} \frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

since the range of integration over τ may be as restricted as much as we like. We have therefore shown that the conservation of the energy-momentum tensor in the presence of a gravitational field (as represented by the geometry of space-time), which follows from the Bianchi identity

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\,\nu} = 0\ ,$$

is equivalent to Newton's Second Law with gravitation treated as an external force. It is this possibility that has led J.A. Wheeler to coin the term "geometrodynamics"----and that led Einstein in the last years of his life to seek a geometrical description of all of physics.