Gravitational radiation

Reading: Ohanian and Ruffini, *Gravitation and Spacetime*, 2nd ed., Ch. 5.

Gravitational equations in empty space

The linearized field equations are (in our units, not Ohanian's)

$$\partial_\lambda \partial^\lambda \varphi^{\mu\nu} = -16\pi G\, T^{\mu\nu} \tag{28.1}$$

In vacuo, $T^{\mu\nu} = 0$, hence

$$\partial_\lambda \partial^\lambda \varphi^{\mu\nu} = 0\,, \tag{28.2}$$

with the gauge condition

$$\partial_\mu \varphi^{\mu\nu} = 0\,. \tag{28.3}$$

Clearly Eq. 28.3 has plane-wave solutions

$$\varphi^{\mu\nu} = \varepsilon^{\mu\nu} e^{ik\cdot x}\,. \tag{28.4}$$

As a consequence of Eq. 28.2, we have

$$k_\mu k^\mu = 0\,; \tag{28.5}$$

and of Eq. 28.3, that

$$k_\mu \varepsilon^{\mu\nu} = 0\,. \tag{28.6}$$

The dispersion relation, Eq. 28.5, describes waves propagating at the speed of light, independent of their frequency $\omega \equiv k^0$.

The polarization tensor

Clearly, $\varepsilon^{\mu\nu}$ is symmetric, hence has 10 components. The 4 conditions 28.6 mean only 6 of the components are independent. But in fact, there are only 2 independent components.

Why is this? We can always make a change of coordinates

$$\bar{x}^\mu \leftarrow x^\mu + \zeta^\mu(x) \tag{28.7}$$

in which case the metric changes from

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{28.7}$$

to

$$\bar{g}^{\mu\nu} = \eta^{\mu\nu} + \bar{h}^{\mu\nu} \tag{28.8}$$

where

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \partial^\nu \zeta^\mu - \partial^\mu \zeta^\nu\,. \tag{28.9}$$

Exercise:
Derive Eq. 28.9.

With the change ov variable Eq. 28.7, we see that $\varphi^{\mu\nu} \to \overline{\varphi}^{\mu\nu}$ where

$$\overline{\varphi}^{\mu\nu} = \varphi^{\mu\nu} - \partial^{\nu} \zeta^{\mu} - \partial^{\mu} \zeta^{\nu} + \eta^{\mu\nu} \partial_{\lambda} \zeta^{\lambda} . \tag{28.10}$$

The new field function obeys the gauge condition $\partial_{\mu} \overline{\varphi}^{\mu\nu} = 0$ if and only if the coordinate transformations are solutions of

$$\partial_{\lambda} \partial^{\lambda} \zeta^{\mu} = 0 . \tag{28.11}$$

Thus, for example, we could modify the solution 28.4 by adding

$$\delta \varphi^{\mu\nu} = \left(a^{\mu} k^{\nu} + a^{\nu} k^{\mu} - \eta^{\mu\nu} k_{\lambda} a^{\lambda} \right) e^{ik \cdot x} \tag{28.12}$$

without changing any physical consequences. Since a^{μ} is an arbitrary constant vector, there are four (4) more conditions on $\varepsilon^{\mu\nu}$, leaving only 2 non-trivial components.

We see that the polarization tensor can be transformed by

$$\overline{\varepsilon}^{\mu\nu} = \varepsilon^{\mu\nu} + a^{\mu} k^{\nu} + a^{\nu} k^{\mu} - \eta^{\mu\nu} k_{\lambda} a^{\lambda} , \tag{28.13}$$

which automatically satisfies

$$k_{\mu} \overline{\varepsilon}^{\mu\nu} = 0 . \tag{28.14}$$

Suppose we have two distinct normalized vectors A_{μ} , B_{μ} such that

$$A_{\mu} A^{\mu} = B_{\mu} B^{\mu} = -1 ,$$

and we make a symmetric tensor out of them *via*

$$\varepsilon^{(1)}_{\mu\nu} \overset{df}{=} A_{\mu} B_{\nu} + A_{\nu} B_{\mu} \tag{28.15}$$

Then, clearly, to satisfy the guage condition, we must have

$$k_{\mu} A^{\mu} = k_{\mu} B^{\mu} = 0 .$$

There are three independent vectors that satisfy this, for a given direction of $\vec{k}$. Call $\hat{k}$ the z-direction---then since $k^{\mu} = (k, 0, 0, k)$, we have

$$A_{\mu} = \begin{cases} (0, -1, 0, 0) \\ (0, 0, -1, 0) \\ \frac{1}{\sqrt{2}} (1, 0, 0, -1) = \frac{1}{k\sqrt{2}} k_{\mu} \end{cases} \tag{28.16}$$

We see that if we pick---say---B_{μ} to be longitudinal, *i.e.* proportional to k_{μ} , then we can always find a transformation like Eq. 28.13 that will reduce $\varepsilon^{(1)}_{\mu\nu}$ to zero. Thus we only can use the *transverse* components $(0, -1, 0, 0)$ and $(0, 0, -1, 0)$. We write

$$A_{\mu} = (0, -1, 0, 0) \tag{28.17a}$$

and

$$B_{\mu} = (0, 0, -1, 0) . \tag{28.17b}$$

Similarly, we can construct a second symmetric tensor

$$\varepsilon^{(2)}_{\mu\nu} = \alpha A_\mu A_\nu + \beta B_\mu B_\nu \tag{28.18}$$

that cannot be reduced to zero by Eq. 28.13. But what are α and β?

One obvious aspect of $\varepsilon^{(1)}_{\mu\nu}$ is what happens to it when we transform the coordinates by a rotation about the z-axis:

$$\begin{aligned} A &\to A\cos\theta + B\sin\theta \\ B &\to -A\sin\theta + B\cos\theta \end{aligned} \tag{28.19}$$

$$\varepsilon^{(1)}_{\mu\nu} \to \varepsilon^{(1)}_{\mu\nu}\cos(2\theta) - \left(A_\mu A_\nu - B_\mu B_\nu\right)\sin(2\theta) \tag{28.20}$$

Equation 28.20 suggests taking $\alpha = 1$, $\beta = -1$ in 28.19. We can test this by examining what happens to e under rotation by an angle θ about the z-axis:

$$\varepsilon^{(2)}_{\mu\nu} \to \varepsilon^{(1)}_{\mu\nu}\sin(2\theta) + \varepsilon^{(2)}_{\mu\nu}\cos(2\theta)\ . \tag{28.21}$$

That is, the two tensors transform into each other under rotations of angle q about the z-axis, exactly like the two orthogonal vectors in Eq. 28.19.

The fact that the coefficients involve $\cos(2\theta)$ and $\sin(2\theta)$ implies that the angular momentum carried by a gravitational plane wave quantum is $2\hbar$. It is worth reviewing briefly how we make this identification. Recall that the (4-vector) solution of Maxwell's equations is

$$A_\mu = a_\mu e^{ik\cdot x}$$
$$k_\mu k^\mu = 0$$
$$k_\mu a^\mu = 0\ .$$

Gauge invariance means we can always add to A_μ a function $\partial_\mu \Lambda(x)$ without changing anything. Thus there is no significance to $a_\mu \propto k_\mu$, and we take a_μ to be A_μ or B_μ above. Then a rotation about the $\hat{k}$ axis involves $\cos(\theta)$ and $\sin(\theta)$ as in Eq. 28.19. We can always write a rotation matrix about the axis R in the form

$$\mathsf{R}(\theta) = e^{i\vec{J}\cdot\hat{n}\theta} \tag{28.22}$$

where the 3 matrices $\vec{J}$ represent the angular momentum operator (in units of $\hbar$). Thus, the effect of Eq. 28.22 operating on a state with definite z-component $J_z = m$ is to multiply the state by a phase $e^{im\theta}$.

Finally, it can be shown† that massless particles with spin $\hbar S$ can have only 2 spin states, with

† This contrasts with the $2S+1$ states expected for a massive particle of spin S. We have just shown this is the case when $S=1$ and $S=2$ (photon and graviton).

$$m \equiv J_z = \pm S\,.$$

Thus the electromagnetic field, being a massless vector field, has $S = 1$ because the rotation of its polarization vectors involves $\cos(\theta)$ and $\sin(\theta)$, and we therefore conclude that gravitational quanta naturally carry spin 2 because of the appearance of 2θ in the rotations about the propagation vector.

Gravitational radiation by a source

We recall the dynamical equation of gravitational radiation, in the weak-field limit:

$$\partial_\lambda \partial^\lambda \varphi^{\mu\nu} = -16\pi G\, T^{\mu\nu} \qquad (28.1)$$

for which the (retarded) solution is (see Eq. 9.8 and the equations leading up to it)

$$\varphi^{\mu\nu}(x) = \int d^3x' \int_{-\infty}^{t} dt' \,\frac{T^{\mu\nu}(\vec{x}', t')}{|\vec{x}' - \vec{x}|}\,\delta(|\vec{x}' - \vec{x}| - t + t')$$

$$= \int d^3x' \,\frac{T^{\mu\nu}(\vec{x}', t - |\vec{x}' - \vec{x}|)}{|\vec{x}' - \vec{x}|} \qquad (28.23)$$

Now, suppose $T^{\mu\nu}$ is a localized, time-varying distribution of matter and energy. Then it of course obeys the conservation equation

$$T^{\mu\nu}{}_{,\nu} = 0\,. \qquad (28.24)$$

Moreover, we are interested----as in the electromagnetic case----in the gravitational field far from the source(s), *i.e.* for $|\vec{x}| >> |\vec{x}'|$. Clearly, there can be no outgoing radiation across a sphere of radius R large compared with the extension of the source, from terms of the (distant) field that fall off faster than $1/R$.

Thus we can replace $|\vec{x} - \vec{x}'|$ with $|\vec{x}| = R$ in the denominator of Eq. 28.23, and are left with evaluating the integral

$$S^{\mu\nu} = \int d^3x' \; T^{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|)\,. \qquad (28.25)$$

The easiest way to deal with the integral is to write

$$T^{\mu\nu}(\vec{x}', t) = \int_{-\infty}^{+\infty} d\omega \; \widetilde{T}^{\mu\nu}(\vec{x}', \omega) \qquad (28.26)$$

so that, using

$$|\vec{x} - \vec{x}'| \approx R - \vec{x}' \cdot \hat{x}\,,$$

and defining the propagation vector of the outgoing radiation

$$\vec{k} \overset{df}{=} \omega\,\hat{x}$$

(hat is, if we are observing gravitational radiation at point $\vec{x}$, the waves that get into our detector propagate in the $\hat{x}$ direction) we find

$$S^{\mu\nu} = \int_{-\infty}^{+\infty} d\omega\, e^{i\omega(R-t)} \int d^3x' \; \widetilde{T}^{\mu\nu}(\vec{x}', \omega)\, e^{-i\vec{k}\cdot\vec{x}'}\,. \qquad (28.27)$$

Now we make another approximation, the so-called *long-wavelength limit*. If the wavelength of the gravitational radiation is long compared with the source, we may replace the factor

$$e^{-i\vec{k}\cdot\vec{x}\,'}$$

appearing in Eq. 28.27 by unity. For a source such as two compact objects with masses of the order $M_\odot$ in an orbit whose closest approach is the solar radius, 700,000 Km (very close indeed!) it is easy to see that

$$kR \equiv \frac{\omega}{c} R \approx \left(\frac{GM}{c^2}\right)^{1/2} \approx 10^{-3}$$

hence the replacement is amply justified.

We see that the only non-trivial polarization components are $(\mu\nu) = (jk)$, where the j,k directions are orthogonal to $\vec{k}$. Thus we are interested in T^{jk}. The conservation equation becomes

$$-i\,\omega\,\widetilde{T}^{0\nu} + \partial_j\,\widetilde{T}^{j\nu} = 0 \tag{28.28}$$

and, moreover,

$$\partial_j\,\partial_k\,\widetilde{T}^{jk} = -\omega^2\,\widetilde{T}^{00} \tag{28.29}$$

leading to

$$S^{jk} \approx \int_{-\infty}^{+\infty} d\omega\; e^{i\omega(R-t)} \int d^3\vec{x}\,'\;\; \widetilde{T}^{\mu\nu}(\vec{x}\,', \omega)\,. \tag{28.30}$$

We can rewrite Eq. 28.30 in a useful way: multiply Eq. 28.29 with $x'^r\,x'^s$ and integrate over x' : we find

$$\int d^3x'\;\; x'^r\,x'^s\;\partial_j\,\partial_k\,\widetilde{T}^{jk}(x', \omega) = -\omega^2 \int d^3x'\;\; x'^r\,x'^s\;\;\widetilde{T}^{00}(x', \omega) \tag{28.31}$$

which can be integrated by parts twice using the identity

$$x^r\,\frac{\partial}{\partial x^k} f(x) \equiv \frac{\partial}{\partial x^k}\left(x^r f(x)\right) - \delta^r_k\, f(x) \tag{28.32}$$

to get

$$\int d^3x'\;\;\widetilde{T}^{rs}(x', \omega) = -\,\tfrac{1}{2}\,\omega^2 \int d^3x'\;\; x'^r\,x'^s\;\;\widetilde{T}^{00}(x', \omega)\,. \tag{28.33}$$

Eq. 28.33 now lets us write the field in the wave zone, in the long wavelength limit, as

$$\varphi^{rs}(\vec{x}, t) = -\,\tfrac{4}{3}\,\frac{G\,\ddot{Q}(t-R/c)}{R} \tag{28.34}$$

where the $-\omega^2$ factor in Eq. 28.32 translates to the second time derivative, and where we have defined the (retarded) *quadrupole moment* of the energy density of the source:

$$Q^{rs}(t-R/c) \stackrel{df}{=} \int d^3x\left(\tfrac{3}{2}\,x^r\,x^s + \tfrac{1}{2}\,r^2\,\eta^{rs}\right)\widetilde{T}^{00}(x, t-R/c)\,. \tag{28.35}$$

The astute student may be wondering, "Who ordered the term $\frac{1}{2} r^2 \eta^{rs}$ appearing in the quadrupole moment?" As we have already seen, a polarization tensor proportional to $\eta^{\mu\nu}$ carries away no flux. Therefore we can add such a term to the polarization tensor---but of course we want to add it with the appropriate coefficient to give the quadrupole moment.

We now need to compute the flux of energy carried off by the gravitational wave. If $t^{\mu\nu}$ is the energy-momentum tensor of the gravitational field, we see from---say---Ohanian, Eq. 4.62†, that

$$t_{\mu\nu} = \frac{1}{16\pi G}\left[2\varphi^{\alpha\beta}{}_{,\mu}\,\varphi_{\alpha\beta\,,\nu} - \varphi_{,\mu}\,\varphi_{,\nu} - \eta_{\mu\nu}\left(\varphi^{\alpha\beta,\lambda}\,\varphi_{\alpha\beta,\lambda} - \tfrac{1}{2}\,\varphi_{,\lambda}\,\varphi^{,\lambda}\right)\right]. \quad (28.36)$$

Hence the flux across a surface-element of a sphere at $\vec{x} = R\,\hat{k}$ is

$$t^{j0}\,\hat{k}_j\,R^2\,d\Omega = \frac{1}{16\pi G}\left(2\varphi^{\alpha\beta,0}\,\varphi_{\alpha\beta}{}^{,j} - \varphi_{,0}\,\varphi_{,}{}^{j}\right)\hat{k}_j\,R^2\,d\Omega \quad (28.37)$$

which, for a sinusoidally oscillating source of frequency ω, gives a total radiated power

$$P = \frac{G\,\omega^6}{9\pi}\,Q^{rs}\,Q_{rs}\int d\Omega\left(P_2\,(\cos\theta)\right)^2 = \frac{4G\,\omega^6}{45c^5}\,Q^{rs}\,Q_{rs}\,. \quad (28.38)$$

The difference (a factor 4) from Eq. 5.73 in Ohanian and Ruffini arises entirely from the factor of two difference in the definition of the mass quadrupole tensor. That is, their quadrupoles are twice as large as ours, which follow the standard conventions.

To obtain the above expression we have used

$$\varphi \overset{df}{=} \varphi^{00} - \sum_k \varphi^{kk} = \hat{k}_j\,\hat{k}_k\,\varphi^{jk},$$

and

$$\varphi^{0k} = \hat{k}_j\,\varphi^{jk}$$

i.e.

$$2\varphi^{\alpha\beta,0}\,\varphi_{\alpha\beta,j} - \varphi_{,0}\,\varphi_{,j} = \left(\dot{\varphi}^{00}\,\dot{\varphi}^{00} - 4\sum_k \dot{\varphi}^{0k}\,\dot{\varphi}^{0k} + 2\sum_{k,l}\dot{\varphi}^{kl}\,\dot{\varphi}^{kl}\right)\hat{k}_j$$

$$= \left(\hat{k}_a\,\hat{k}_b\,\hat{k}_r\,\hat{k}_s - 4\delta_{ar}\,\hat{k}_b\,\hat{k}_s + 2\delta_{ar}\,\delta_{bs}\right)\dot{\varphi}^{ab}\,\dot{\varphi}^{rs}\,\hat{k}_j\,.$$

The averages of the propagation vectors over solid angle are easily seen to be

$$\frac{1}{4\pi}\int d\Omega\;\hat{k}_a\,\hat{k}_b = \tfrac{1}{3}\,\delta_{ab}$$

and

$$\frac{1}{4\pi}\int d\Omega\;\hat{k}_a\,\hat{k}_b\,\hat{k}_r\,\hat{k}_s = \tfrac{1}{15}\left(\delta_{ab}\,\delta_{rs} + \delta_{ar}\,\delta_{bs} + \delta_{as}\,\delta_{br}\right).$$

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† Note the difference in normalization—this comes from the way we have defined the Lagrangian.