Cosmology

Reading: Weinberg, Ch. 11.

A metric tensor appropriate to infalling matter

In general (see, *e.g.*, Weinberg, Ch. 11) we may write a spherically symmetric, time-dependent metric in the form

$$(d\tau)^2 = B(r,t)\,(dt)^2 - A(r,t)\,(dr)^2 - r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\varphi)^2 \qquad (29.1)$$

and from this we deduce

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4}\left(\frac{B'}{A}\right)\left(\frac{A'}{A}+\frac{B'}{B}-\frac{1}{r}\right) + \frac{\ddot{A}}{2A} - \frac{1}{4}\left(\frac{\dot{A}}{A}\right)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right) \qquad (29.2t)$$

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\left(\frac{B'}{B}\right)\left(\frac{A'}{A}+\frac{B'}{B}\right) - \frac{A'}{rA} + \frac{\ddot{A}}{2B} - \frac{1}{4}\left(\frac{\dot{A}}{B}\right)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right) \qquad (29.2r)$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(\frac{B'}{B}-\frac{A'}{A}\right) + \frac{1}{A} \qquad (29.2\theta)$$

$$R_{\varphi\varphi} = R_{\theta\theta}\,\sin^2\theta \qquad (29.2\varphi)$$

$$R_{tr} \equiv R_{rt} = -\frac{\dot{A}}{rA} \qquad (29.3)$$

with all other terms = 0.

We now want to solve the Einstein equations in the following cases:

1. Empty space

Equation 29.3 implies that

$$\dot{A} = \ddot{A} = 0\,,$$

hence A is time-independent. By the methods used earlier, we find†

$$AB = 1\,,$$

therefore B is time-independent also. Thus we recover the Schwarzschild metric,

$$A(r) = \left(1 - \frac{2MG}{r}\right)^{-1} \qquad (29.4)$$

† This reflects a specific choice of time coordinate, as before.

This proves *Birkhoff's Theorem*----a spherically symmetric metric in empty space is time-independent. Thus there can be no gravitational radiation from a spherical source.

2. Dust to dust

Let us redefine r and t to get co-moving coordinates, appropriate to an oberver falling freely with some particular piece of matter:

$$(d\tau)^2 = (dt)^2 - U(r,t)\,(dr)^2 - V(r,t)\left[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right] \tag{29.5}$$

then with this metric,

$$R_{tt} = \frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{1}{4}\frac{\dot{U}^2}{U^2} - \frac{1}{2}\frac{\dot{V}^2}{V^2} \tag{29.6t}$$

$$R_{rr} = \frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)\left(\frac{U'}{U} + \frac{V'}{V}\right) - \frac{1}{2}\ddot{U} + \frac{\dot{U}^2}{4U} - \frac{\dot{U}\,\dot{V}}{2V} \tag{29.6r}$$

$$R_{\theta\theta} = -1 + \frac{V''}{2U} - \frac{U'\ V'}{4U^2} - \frac{1}{2}\ddot{V} - \frac{\dot{U}\,\dot{V}}{4U} \tag{29.6θ}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\ \sin^2\theta \tag{29.6φ}$$

$$R_{tr} = \frac{\dot{V}'}{V} - \frac{\dot{V}\,V'}{2V^2} - \frac{\dot{U}\,V'}{2UV} \tag{29.6tr}$$

and all other components vanish.

Dust can be defined as a group of particles with energy density but no pressure. An example is a large cluster of well-spaced galaxies. The energy-momentum tensor of dust is

$$T^{\mu\nu} = \rho\, U^{\mu}\, U^{\nu} \tag{29.7}$$

The invariant volume element is

$$dt\, dr\, d\theta\, d\varphi\, \sqrt{g} = dt\, dr\, d\theta\, d\varphi\, V \sin\theta\, \sqrt{U} \tag{29.8}$$

In co-moving coordinates, there is no local motion of a particle, hence

$$U^{\mu} = \begin{pmatrix} U^{t} \\ U^{r} \\ U^{\theta} \\ U^{\varphi} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Of the 4 equations

$$T^{\mu\nu}{}_{;\nu} = 0\,, \tag{29.9}$$

the space components are automatically satisfied:

$$T^{k\nu}{}_{;\nu} = 0 \tag{29.10}$$

leaving

$$T^{0\nu}{}_{;\nu} = 0 \tag{29.11}$$

or

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}(T^{t\nu}\sqrt{g}) + \begin{Bmatrix} t \\ \sigma\,\nu \end{Bmatrix} T^{\sigma\nu} = \frac{1}{V\sqrt{U}}\frac{\partial}{\partial t}(\rho\, V\sqrt{U}) + \rho \begin{Bmatrix} t \\ t\ t \end{Bmatrix} = 0 \tag{29.12}$$

and since

$$\begin{Bmatrix} t \\ t\ t \end{Bmatrix} = \tfrac{1}{2} g^{tt}\,\partial_t g_{tt} = 0\,,$$

we have

$$\frac{\partial}{\partial t}(\rho\, V\sqrt{U}) = 0\,. \tag{29.13}$$

The gravitational field equations become

$$R_{tt} = \frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{1}{4}\frac{\dot{U}^2}{U^2} - \frac{1}{2}\frac{\dot{V}^2}{V^2} = -4\pi G\rho \tag{29.14t}$$

$$R_{rr} = \frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)\left(\frac{U'}{U} + \frac{V'}{V}\right) - \frac{1}{2}\ddot{U} + \frac{\dot{U}^2}{4U} - \frac{\dot{U}\,\dot{V}}{2V} = -4\pi G\rho \tag{29.14r}$$

$$R_{\theta\theta} = -1 + \frac{V''}{2U} - \frac{U'\ V'}{4U^2} - \frac{1}{2}\ddot{V} - \frac{\dot{U}\,\dot{V}}{4U} = -4\pi G\rho \tag{29.14θ}$$

$$R_{\varphi\varphi} = R_{\theta\theta}\ \sin^2\theta = -4\pi G\rho\,\sin^2\theta \tag{29.14φ}$$

$$R_{tr} = \frac{\dot{V}'}{V} - \frac{\dot{V}\,V'}{2V^2} - \frac{\dot{U}\,V'}{2UV} = 0\,. \tag{29.14tr}$$

To solve these, multiply Eq. 29.14*tr* by V and divide by V', so

$$\frac{d}{dt}\ln(V') - \frac{1}{2}\frac{d}{dt}\ln(UV) = 0$$

or

$$V' = F(r)\sqrt{UV} \tag{29.15}$$

where $F(r)$ is some arbitrary---for the moment---function of r.

Next add Eq. 29.14*r* and Eq. 29.14*t* and subtract twice Eq. 29.14θ, to get

$$-\frac{V'^{\,2}}{2UV^2} + \frac{2}{V} + \frac{2\ddot{V}}{V} - \frac{\dot{V}^2}{2V^2} = 0\ . \tag{29.16}$$

Combining Eq. 29.16 with Eq. 29.15 we get

$$-\frac{1}{2}F^2(r) + 2 + 2\ddot{V} - \frac{\dot{V}^2}{2V} = 0\,.$$

We see that $2\ddot{V} - \dfrac{\dot{V}^2}{2V}$ is independent of *t*. This suggests

$$V(r, t) = R^2(t)\ \Gamma(r),$$

and thence

$$2\,\Gamma(r)\left(2\dot{R}^2 + 2\dot{R}\ddot{R} - \dot{R}^2\right) - \tfrac{1}{2}F(r) + 2 = 0 \tag{29.17}$$

i.e.,

$$\dot{R}^2 + 2\dot{R}\ddot{R} = \text{constant} \overset{df}{=} -k\,. \tag{29.18}$$

From Eq. 29.15 we have

$$\Gamma'(r)\,R(t) = F(r)\sqrt{\Gamma(r)\,U} \tag{29.19}$$

which suggests

$$U(r, t) = R(t)\,f(r).$$

We are at liberty to redefine *r* so that $\Gamma(r) = r^2$; thus from Eq. 29.17

$$F^2(r) = 4\,(1 - kr^2) \tag{29.20}$$

and from Eq. 29.19

$$f(r) = \frac{4}{F^2(r)} = (1 - kr^2)^{-1} \tag{29.21}$$

The metric now has the form

$$(d\tau)^2 = (dt)^2 - R^2(t)\left[\frac{(dr)^2}{1 - kr^2} + r^2\left((d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right)\right]. \tag{29.22}$$

This is called the ***Robertson-Walker*** metric.

We suppose that the density varies with time only (since the radial and angular velocity components vanish). Then energy conservation becomes

$$\frac{\partial}{\partial t}\left(\rho\,V\sqrt{U}\right) = r^2\sqrt{f(r)}\,\frac{\partial}{\partial t}\left(\rho(t)\,R^3(t)\right) = 0 \tag{29.23}$$

from which we deduce

$$\rho(t) = \rho(0)\,\frac{R^3(0)}{R^3(t)}\,. \tag{29.24}$$

Time evolution of dust

By convention, $R(0) = 1$. From Eq. 29.14*t*,

$$R_{tt} = \frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{1}{4}\frac{\dot{U}^2}{U^2} - \frac{1}{2}\frac{\dot{V}^2}{V^2} = -4\pi G\rho \ ,$$

we find

$$3\frac{\ddot{R}}{R} = -4\pi G\rho(t) = -\frac{4\pi G\rho(0)}{R^3(t)} \qquad (29.25)$$

We also had

$$\dot{R}^2 + 2R\ddot{R} = -k$$

which when combined with Eq. 29.25 gives

$$k + \dot{R}^2 = \frac{8\pi G\rho(0)}{3R(t)} \qquad (29.26)$$

If $\dot{R}(0) = 0$, then we may re-write Eq. 29.26 as

$$\dot{R}(t) = -\left(\frac{8\pi G\rho(0)}{3}\right)^{1/2}\left(\frac{1}{R(t)} - 1\right)^{1/2} \qquad (29.27)$$

where we choose the negative root in order to describe collapse. That is, the dust, initially in some distribution with $\rho(0) \neq 0$, falls freely inward. To simplify Eq. 29.27, let

$$R(t) = \frac{1}{2}(1 + \cos\psi) \ .$$

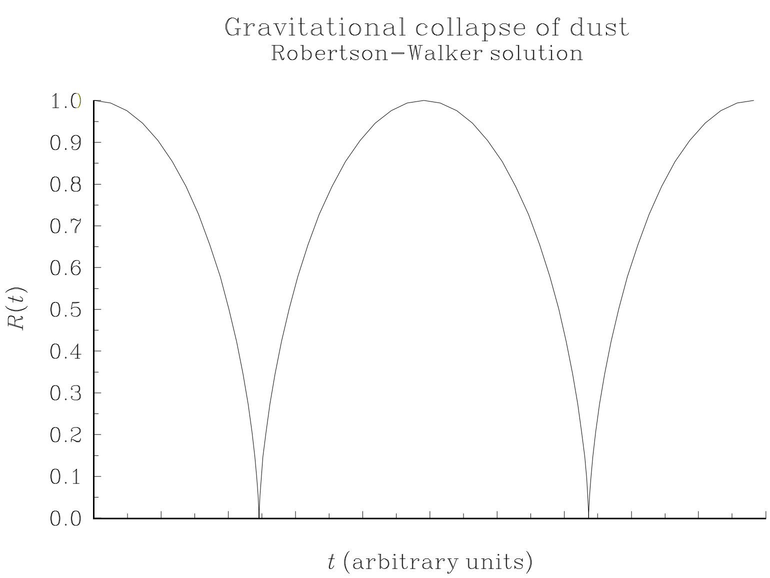
Then

$$\frac{1}{2}\dot{\psi}\sin\psi = \lambda\left(\frac{1 - \cos\psi}{1 + \cos\psi}\right)^{1/2} \equiv \lambda\frac{\sin\psi}{1 + \cos\psi} \qquad (29.28)$$

which can be integrated simply to give (note $\psi(0) = 0$)

$$\frac{1}{2}(\psi + \sin\psi) = \lambda t \ . \qquad (29.29)$$

Eq. 29.29 describes a cycloid:



The time to collapse is evidently

$$T_{collapse} = \frac{\pi}{2\lambda} \qquad (29.30)$$

Since, if $k>0$, the Robertson-Walker metric demands

$$r^2 < \frac{1}{k} = \frac{3}{8\pi G\rho(0)}$$

we see that the collapse time is

$$T = \frac{\pi}{2} \times \frac{r_{\max}}{c} .$$

To recapitulate, a ball of dust (*i.e.* $p=0$), initially at rest, will collapse to a point, under its mutual gravitational attraction, in time T.

Outside the ball of dust

It is possible to put the metric outside in Schwarzschild form

$$(d\tau)^2 = B(r')\,(dt')^2 - A(r')\,(dr')^2 - r'^2\left((d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right) \qquad (29.31)$$

To do this, we need to match up at the surface. Let $\theta' = \theta$, $\varphi' = \varphi$ and $r' = r\,R(t)$. Then

$$dr' = R\,dr + r\,\dot{R}\,dt$$

and we define the "outside" time by†

$$t' = \left(\frac{1-ka^2}{k}\right)^{1/2} \int_{S(r,\,t)}^{1} \frac{dR}{1 - ka^2/R} \left(\frac{R}{1-R}\right)^{1/2} \qquad (29.32a)$$

$$S(r, t) = 1 - \left(\frac{1-kr^2}{1-ka^2}\right)^{1/2} (1 - R(t)) . \qquad (29.32b)$$

If we fit at $r = a$ (the radius of the dustball) then we have

$$B(a, t') = 1 - \frac{ka^3}{aR(t)} \qquad (29.33)$$

$$A(a, t') = \left(1 - \frac{ka^3}{aR(t)}\right)^{-1} . \qquad (29.34)$$

This matches the outside solution if $2MG = ka^3$. But from Eq. 29.26 we have

$$k = \frac{8\pi G\rho(0)}{3}$$

so the condition is

$$M = \frac{4\pi a^3}{3}\,\rho(0) ,$$

----not a very surprising result!

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† see Weinberg, p. 345.

Collapse seen by outside observer

We now ask what the collapse looks like to a distant observer. We see that if light is emitted radially from the surface of the star at (outside) time t_0, it propagates according to

$$d\tau = 0,$$

or

$$dt' = A(r')\, dr'$$

hence

$$t' = t_0 + \int_{a\,R(t)}^{r'} dr \left(1 - \frac{2MG}{r}\right)^{-1} \tag{29.35}$$

We see that as $R(t)$ approaches $2MG/a$ (that is, as the surface approaches the Schwarzschild radius, the time for the light to reach the observer becomes (logarithmically) *infinite*. The gravitational red shift of light reaching the observer becomes

$$z \overset{df}{=} \frac{dt'}{dt} - 1 = \frac{dt_0}{dt} - a\,\dot{R}(t)\left(1 - \frac{2MG}{a\,R(t)}\right)^{-1} - 1 \tag{29.36}$$

hence as the radius reaches r_S,

$$z \to \exp\left(\frac{t'}{2MG}\right). \tag{29.37}$$

For most of the star's life, $r >> r_S$ and $t' \approx t$, *i.e.* the redshift is essentially zero. But as the end of the collapse approaches, an outside observer sees an exponentially increasing redshift, *i.e.* the star disappears into redness, with a time scale of minutes. The further collapse to $R(T) = 0$ is invisible to an outside observer.

A co-moving observer has no difficulty† seeing the collapse to $R = 0$. His time becomes disconnected from that of the outside world after he passes within the Schwarzschild radius. The surface $r' = r_S$ represents a trapped discontinuity that separates inside from outside. Stuff can fall in, but it can never get out again, in classical General Relativity.

Model universes

The most appropriate metric for cosmology is the spatially homogeneous Robertson-Walker metric

$$(d\tau)^2 = (dt)^2 - R^2(t)\left[\frac{(dr)^2}{1 - kr^2} + r^2\left((d\theta)^2 + \sin^2\theta\,(d\varphi)^2\right)\right] \tag{29.22}$$

that arises automatically from co-moving coordinates. The Robertson-Walker metric embodies the idea that at fixed t (spacelike hypersurface) any point is equivalent to any other point. The curvature of the 3-dimensional hypersurfaces $t =$ const. is $K_3(t) = k\,R^{-2}(t)$. By rescaling r and $R(t)$ k can be

† …assming tidal forces do not exceed his personal Roché limit.

normalized to ± 1, if $k \neq 0$. Thus, a space of positive curvature K_3 kas $k = +1$, and a space of negative curvature has $k = -1$.

Positive curvature

When $k = +1$, the proper circumference of the space is

$$L_3 = 2\pi R(t) \tag{29.38a}$$

and the proper volume is

$$V_3 = 2\pi^2 R^3(t) \ . \tag{29.38b}$$

At fixed t the universe is the surface of a 3-sphere of radius $R(t)$ embedded in a Euclidean 4-dimensional manifold, so $R(t)$ is the "radius" of the universe.

Space is *finite*, but *unbounded* (since $(dr)^2/(1-kr^2) \to \infty$).

Zero curvature

When $k = 0$, we say space is *flat* (in an average or global sense). Flat space is infinite, since the 3-dimensional hypersurfaces $t =$ const. are open.

Negative curvature

When $k = -1$ space is also infinite because a negatively curved hypersurface

$$K_3 = \text{const.} < 0$$

is open.

Influence of matter

In isotropic 3-space T^{00} must be scalar with respect to transformations of r, θ , φ ; hence

$$T^{00} = \rho(t) \tag{29.39a}$$

$$T^{k0} = 0 \tag{29.39b}$$

$$T^{jk} = -p(t)\, g^{jk} \tag{29.39c}$$

We can define a flux of galaxies J_G^μ :

$$J_G^0 = n_G(t) \tag{29.40a}$$

$$J_G^\mu = n_G U^\mu \tag{29.40b}$$

and then

$$T^{\mu\nu} = -p\, g^{\mu\nu} + (p+\rho)\, U^{\mu} U^{\nu} \tag{29.41}$$

and as before,

$$U^0 = 1$$

$$U^k = 0\ .$$

Conservation of galaxies may be written

$$g^{-1/2} \frac{\partial}{\partial t}\left(g^{1/2}\, n_G\right) = 0 \tag{29.42}$$

and with

$$g = R^6(t)\, \frac{r^4 \sin^2\theta}{1 - kr^2}$$

we find, unsurprisingly,

$$n_G(t)\; R^3(t) = \text{const.} \tag{29.43}$$

Conservation of energy-momentum,

$$T^{\mu\nu}{}_{;\nu} = 0$$

implies

$$R^3(t)\, \frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left[R^3(t)\,(p + \rho)\right]\ . \tag{29.44}$$

If pressure is negligible, then as for the "dust" model of a collapsing star,

$$\rho(t)\, R^3(t) = \text{const.} \tag{29.45}$$

Note that 29.45 and 29.43 are inequivalent unless we neglect pressure.

Proper distances

Imagine observers in galaxies along a line of sight to some distant galaxy at r_n , at some cosmic time t. Each measures the distance to the next galaxy by----say----the travel time for a light signal. Then the sum of the distances along the line of sight would be

$$\sum_n ds_n = \sum_n \left(\frac{(dr_n)^2\, R^2(t)}{1 - kr^2}\right)^{1/2} \tag{29.46}$$

or if we assume the observers closely spaced relative to the overall distance,

$$D_{proper}(t) \overset{df}{=} \int_0^{r_1} dr\,\left(g_{rr}\right)^{1/2} = R(t)\int_0^{r_1} dr\left(1 - kr^2\right)^{-1/2}\ . \tag{29.47}$$

Cosmic red shift

The equation of motion of light is $d\tau = 0$, or

$$dt = R(t)\,\frac{dr}{\sqrt{1 - kr^2}} \tag{29.48}$$

hence if the light leaves r_1 at t_1 and arrives at $r=0$ (Earth) at time t_0 we have

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = f(r_1)\ . \tag{29.49}$$

The right side of Eq. 29.49 is independent of time. For nearby galaxies, $kr^2 << 1$ so $f(r_1) \approx r_1$.

Assume the next wave crest leaves at $t_1 + \delta t_1$ and arrives at $t_0 + \delta t_0$; then

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{R(t)} = \int_{t_1}^{t_0} \frac{dt}{R(t)} = f(r_1) \tag{29.50}$$

so

$$\frac{\delta t_0}{R(t_0)} - \frac{\delta t_1}{R(t_1)} = 0\ . \tag{29.51}$$

But since the time between successive wave crests, at a fixed location, is

$$\delta t \stackrel{df}{=} \frac{1}{\nu} = \frac{\lambda}{c}\ ,$$

we may write

$$\frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}$$

and thus the red-shift z defined in 29.36 above becomes

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{R(t_0) - R(t_1)}{R(t_1)}\,. \tag{29.52}$$

In an expanding universe, $t_1 < t_0$ so that $R(t_1) < R(t_0)$ and we get a cosmological *red*-shift, $z > 0$.

For a nearby galaxy, we should say the proper distance is

$$D(t) \approx R(t)\,r_1$$

and that its radial velocity is therefore

$$v_{rad} = \dot{D}(t) = \dot{R}(t)\,r_1\ . \tag{29.53}$$

But since

$$R(t_0) - R(t_1) \approx \dot{R}(t_0)\left(t_0 - t_1\right)\,,$$

and

$$\frac{t_0 - t_1}{R(t_0)} \approx r_1$$

we find

$$z \approx \frac{\dot{R}(t_0)}{R(t_0)}\left(t_0 - t_1\right) \approx \frac{\dot{R}(t_0)}{R(t_0)} r_1 R(t_0) = v_r \tag{29.54}$$

i.e.

$$v_r = \frac{\dot{R}(t_0)}{R(t_0)} D(t_0) \ . \tag{29.55}$$

Thus we may identify $\frac{\dot{R}(t_0)}{R(t_0)}$ as the Hubble constant, H_0.

Deceleration parameter

Assuming the cosmic scale parameter $R(t)$ to be well-enough behaved, we may expand it in Taylor's series, measuring time from the present:

$$R(t) \approx R(t_0)\left(1 + \frac{\dot{R}(t_0)}{R(t_0)} t + \frac{1}{2}\frac{\ddot{R}(t_0)}{R(t_0)} t^2 + \ldots\right) . \tag{29.57}$$

Now, the *deceleration parameter* is defined as

$$-q_0 \overset{df}{=} \frac{\ddot{R}(t_0)}{H_0{}^2 \ R(t_0)}$$

(note that deceleration corresponds to $q_0 > 0$). Equation 29.57 can then be written in standard format

$$R(t) \approx R(t_0)\left[1 + H_0 t - \frac{1}{2} q_0 \left(H_0 t\right)^2 + \ldots\right] . \tag{29.58}$$

Our isotropic model universe satisfies the equations

$$\ddot{R} R = -\frac{4\pi G}{3}(\rho + 3p) R^2 \tag{29.59t}$$

$$\ddot{R} R + 2\dot{R}^2 + 2k = 4\pi G (\rho - p) R^2 \tag{29.59r}$$

and thence

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 , \tag{29.60}$$

and (from energy-momentum conservation)

$$R^3 (t) \frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left[R^3 (t) (p + \rho)\right] .$$

There are two obvious cases:

1. $p << \rho$ (dust)

   then $R^3(t)\,\rho \approx$ const. and from Eq. 29.59*t*,

$$\ddot{R} \propto R^{-2} \tag{29.61}$$

which leads to a power-law behavior for $R(t)$:

$$R \propto t^{2/3} \tag{29.62}$$

2. $p = \frac{1}{3}\rho$ (ultrarelativistic gas)

Now, from Eq. 29.61,

$$\rho R^4 = \text{const.}$$

$$R(t) \propto t^{1/2} \tag{29.63}$$

The (dimensionless) deceleration parameter q_0 can be related to the average density of mass-energy in the universe; hence the question of whether the universe is open $(k \le 0)$ or closed $(k > 0)$ can in principle be answered by measuring q_0 . Unfortunately, while the observational evidence that $q_0 > 0$ is good, we cannot say more than that at present. And recently new evidence has been obtained that may indicate $q_0 < 0$, which would mean the expansion of the universe is accelerating.

From Eq. 29.60, we may obtain

$$\frac{k}{R^2(t_0)} + H_0{}^2 = \frac{8\pi G}{3}\rho(t_0) \tag{29.64}$$

and from Eq. 29.59*t* and the definition of q_0 we find

$$q_0 H_0{}^2 = \frac{4\pi G}{3}\left(\rho_0 + 3p_0\right). \tag{29.65}$$

From Eq. 29.65 and Eq. 29.64 we can then derive an expression for the pressure now:

$$p_0 = -\frac{1}{8\pi G}\left(\frac{k}{R_0{}^2} + H_0{}^2(1 - 2q_0)\right). \tag{29.66}$$

Moreover, from Eq. 29.64 we determine that

$$k = R_0{}^2\frac{8\pi G}{3}\left(\rho_0 - \rho_{crit}\right) \tag{29.67}$$

so that the criterion for closure of the metric is $\rho_0 > \rho_{crit}$.

Moreover, if the present pressure is ≈ 0, then setting Eq. 29.66 to 0 gives

$$\rho_0 \approx 2q_0\rho_{crit} \tag{29.68}$$

hence $q_0 > \frac{1}{2} \Rightarrow \rho_0 > \rho_{crit}$ and consequently, $k > 0$.