Due: Tuesday, 5 May 1998 @ 17:00.

1. A 2-dimensional surface embedded in Euclidean 3-space is parameterized by

$$x = \zeta\cos\varphi$$
$$y = \zeta\sin\varphi$$
$$z = \zeta$$

a) What are the elements of the metric tensor $g_{\alpha\beta}$? (let $x^1 = \zeta$, $x^2 = \varphi$)

Solution:

Manifestly,

$$(ds)^2 = 2(d\zeta)^2 + \zeta^2 (d\varphi)^2 ,$$

so the metric is

$$g_{jk} = \begin{pmatrix} 2 & 0 \\ 0 & \zeta^2 \end{pmatrix}.$$

b) Calculate the scalar curvature R. Is the surface open or closed?

Solution

The inverse of the metric tensor is

$$g^{jk} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{\zeta^2} \end{pmatrix}$$

and the only non-zero Christoffel symbols are thus

$$\left\{ \begin{matrix} \zeta \\ \varphi\ \varphi \end{matrix} \right\} = -\tfrac{1}{2}\zeta , \qquad \left\{ \begin{matrix} \varphi \\ \zeta\ \varphi \end{matrix} \right\} = \frac{1}{\zeta} .$$

The Ricci tensor is

$$R_{\mu\nu} = \partial_\nu \left\{ \begin{matrix} \lambda \\ \mu\,\lambda \end{matrix} \right\} - \partial_\lambda \left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\} + \left\{ \begin{matrix} \sigma \\ \mu\,\lambda \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \sigma\,\nu \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \mu\,\nu \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \sigma\,\lambda \end{matrix} \right\}$$

and we easily see its three elements all vanish. Therefore the scalar curvature vanishes also. The surface is thus open.

c) Why should the answer to b) above be unsurprising?

Solution

The surface described by the above coordinates is a right circular cone. Since a cone can be unfolded and mapped into a plane without tearing or stretching, it is necessarily a flat surface, *i.e.* with zero curvature.

2. Some theorists have suggested that a neutron star can undergo (weak) interactions that transform neutrons to Λ hyperons *via* the reactions

$$n \longleftrightarrow \Lambda + \pi^0$$
$$\Lambda \longleftrightarrow n + \pi^0$$
$$p \longleftrightarrow \Lambda + \pi^+$$
$$\Lambda \longleftrightarrow p + \pi^- \ .$$

Useful information:

$$M_\Lambda = 1115.6\,\mathrm{MeV/c^2}$$
$$M_n = 939.6\ \mathrm{MeV/c^2}$$
$$M_p = 938.3\ \mathrm{MeV/c^2} = 1.67\times10^{-24}\ \mathrm{gm}$$
$$m_{\pi^0} = 135.0\ \mathrm{MeV/c^2}$$
$$m_{\pi^\pm} = 139.6\ \mathrm{MeV/c^2}$$
$$M_\odot = 1.99\times10^{33}\ \mathrm{gm}$$

Assume practically all the neutrons in a neutron star have been converted to Λ's. What is the relative concentration of neutrons and protons to that of Λ's? Take

$$M_{star} = M_\odot$$
$$R = 10\ \mathrm{Km}$$

(**Hint:** What would be the momentum of the emitted nucleon in the decay of Λ?)

Solution

To do this problem in its entirety would be very complicated. We should have to minimize the Gibbs free energy in each volume, holding the baryon number and electric charge densities constant. If y is the local neutron fraction and z the proton fraction, then

$$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0$$

where

$$g = \rho_\Lambda - p_\Lambda + \rho_n - p_n + \rho_p - p_p + \rho_e - p_e \ .$$

Then we would have to integrate the equations of hydrostatic equilibrium to get the total mass and radius of the star, not to mention the net fractions of the various baryonic species.

Hence we simplify the work by assuming the density is constant throughout the star. In that case the relative fractions of the different baryons are independent of position, and we may say

$$n_n = yn$$
$$n_p = n_e = zn$$
$$n_\Lambda = (1 - y - z)\,n$$

where the average baryon number density is

$$n = \frac{k_F^3}{3\pi^2} \ .$$

As a shortcut, we use the conditions that

$$\left((1-y-z)^{2/3} k_F^2 + m_\Lambda^2\right)^{1/2} - m_{\pi^0} \le \left(y^{2/3} k_F^2 + m_n^2\right)^{1/2}$$

and

$$\left((1-y-z)^{2/3} k_F^2 + m_\Lambda^2\right)^{1/2} - m_{\pi^-} \le \left(z^{2/3} k_F^2 + m_p^2\right)^{1/2}$$

(the Λ's will not be able to decay if the neutron and proton Fermi energies are too high).

That is, in a neutron star of 1 solar mass, there are about 10^{57} baryons in a sphere of radius 10 Km, giving $k_F \approx 384$ MeV/c . This is nonrelativistic, so the average kinetic energies are about 78.4 (nucleon) and 66 (lambda) respectively. Then we have

$$78.4\, y^{2/3} \approx 66\,(1-y-z)^{2/3} + 41$$

$$78.4\, z^{2/3} \approx 66\,(1-y-z)^{2/3} + 37.7$$

and we find

$$y \approx z \approx 0.477$$

Thus on the average such a star can have only about 4.5% Λ's.

3. Consider the polarization tensors for a gravitational wave propagating in the z direction:

$$\varepsilon^{(1)}_{\mu\nu} = A_\mu B_\nu + A_\nu B_\mu$$

$$\varepsilon^{(2)}_{\mu\nu} = A_\mu A_\nu - B_\mu B_\nu$$

where

$$A_\mu = \left(0, -1, 0, 0\right)$$

$$B_\mu = \left(0, 0, -1, 0\right) .$$

Find linear combinations $\hat{t_\pm}$ of $\varepsilon^{(1)}_{\mu\nu}$ and $\varepsilon^{(2)}_{\mu\nu}$ that, under rotations about the z-axis by an angle θ, transform as

$$\hat{t_\pm}' = e^{\pm 2i\theta}\,\hat{t_\pm} .$$

Solution

Under rotations about the z-axis,

$$A \rightarrow A\cos\theta + B\sin\theta$$
$$B \rightarrow -A\sin\theta + B\cos\theta ,$$

the polarization tensors transform as

$$\varepsilon^{(1)}_{\mu\nu} \rightarrow \varepsilon^{(1)}_{\mu\nu}\cos(2\theta) - \varepsilon^{(2)}_{\mu\nu}\sin(2\theta)$$

$$\varepsilon^{(2)}_{\mu\nu} \rightarrow \varepsilon^{(1)}_{\mu\nu}\sin(2\theta) + \varepsilon^{(2)}_{\mu\nu}\cos(2\theta)$$

Therefore we must choose new tensors (we omit the indices for simplicity)

$$u = \alpha\,\varepsilon^{(1)} + \beta\,\varepsilon^{(2)} \rightarrow \alpha\left(\varepsilon^{(1)}\cos(2\theta) - \varepsilon^{(2)}\sin(2\theta)\right) + \beta\left(\varepsilon^{(1)}\sin(2\theta) + \varepsilon^{(2)}\cos(2\theta)\right)$$

$$= \left(\alpha\,\varepsilon^{(1)} + \beta\,\varepsilon^{(2)}\right)\cos(2\theta) + \left(\beta\,\varepsilon^{(1)} - \alpha\,\varepsilon^{(2)}\right)\sin(2\theta)$$

$$= u\cos(2\theta) + \left(\beta\,\varepsilon^{(1)} - \alpha\,\varepsilon^{(2)}\right)\sin(2\theta) \equiv u\,e^{2i\theta} .$$

In order for the latter equality to be true, we must have

$$\left(\beta\,\varepsilon^{(1)} - \alpha\,\varepsilon^{(2)}\right)\sin(2\theta) = i\,u\sin(2\theta) \equiv i\left(\alpha\,\varepsilon^{(1)} + \beta\,\varepsilon^{(2)}\right)\sin(2\theta) ,$$

or (from the linear independence of the polarization tensors)

$$\beta = i\,\alpha$$
$$-\alpha = i\,\beta$$

These are in fact the same relation, hence we have

$$u = \alpha\left(\varepsilon^{(1)} + i\,\varepsilon^{(2)}\right)$$

where α is an overall scale, conventionally set to $1/\sqrt{2}$.

The complex conjugate of u transforms as $u^* \rightarrow u^*\,e^{-2i\theta}$.

4. A binary star consists of a neutron star and a black hole, respectively of masses $1.3\,M_\odot$ and $5\,M_\odot$. Their orbital separation is 100 Km, and the orbit is circular. What is the rate of decrease of their orbital separation, due to the loss of energy in gravitational radiation?

Solution

The first question we must answer is whether the radiation can be treated in the long wavelength limit. We see that the angular frequency is (neglecting relativistic effects, which may not be legitimate!)

$$\omega^2 = G\left(M_1 + M_2\right)R^{-3}$$

The total mass is $6.3\,M_\odot$ so if R were the Earth's orbital radius, we would expect a period of $T = 1\,yr/\sqrt{6.3} = 1.26\times10^7\,sec$.

But since the period scales as the 3/2 power of radius, we get 0.007 sec for the actual radius of the orbit. The distance light travels in this time is 2×10^6 meters, hence the stars are travelling "only" 30% of the speed of light. Therefore the time dilation and/or Lorentz contraction effects are "only" 5% effects and can be ignored.

The wavelength of a gravitational wave whose period is 0.007 sec is 20 times the orbital radius, hence the long wavelength approximation applies. Hence we may use the formula

$$P = \frac{4G}{45c^5}\dddot{Q}^{rs}\dddot{Q}_{rs}$$

without modification. The mass quadrupole moments of a point mass m in a circular orbit of radius r are (taking the z-axis perpendicular to the orbital plane)

$$Q^{xx} = \frac{3}{4}mr^2\cos(2\omega t) + const.$$

$$Q^{xy} = \frac{3}{4}mr^2\sin(2\omega t) + const.$$

$$Q^{yy} = -\frac{3}{4}mr^2\cos(2\omega t) + const.$$

with all others 0. For two masses orbiting around each other, with R the distance of separation, we have (ignoring the constant since it contributes nothing to the time derivatives)

$$Q^{xx} = \frac{3}{4}\,\frac{M_1 M_2}{M_1+M_2}R^2\cos(2\omega t)\,, \qquad etc.$$

The radiated power is therefore

$$P = \frac{32}{5}\frac{G\,\omega^6}{c^5}\left(\frac{M_1 M_2}{M_1+M_2}\right)^2 R^4 = \frac{32}{5}\left(\frac{GM_\odot}{Rc^2}\right)^4\times(5+1.3)\times(5\times1.3)^2\times M_\odot c^2\times\frac{c}{R}$$

$$= 1703.5\times(0.0147)^4\times1.8\times10^{47}\,\text{Joules}\times\frac{3\times10^8\,\text{m/s}}{10^5}\,\text{m} = 4.3\times10^{46}\,\text{watts}\ .$$

The rate of change of the orbital radius is therefore

$$\dot{R} = -\frac{2P\,R^2}{GM_1\,M_2} = -1.68\times10^{-3}\,c.$$

5. What is the "proper" distance—as we defined it in class and notes—of a quasar with an apparent recession velocity $0.95c$? Take the Hubble constant to have the value

$$H_0 = 15\times10^{-6}\ \text{km/sec/light-year},$$

and assume the universe to be asymptotically flat with cosmological constant 0.

Solution

The proper distance is given in terms of the radial distance by Eq. 29.47,

$$D_{proper}(t) \overset{df}{=} \int_0^{r_1} dr \left(g_{rr}\right)^{1/2} = R(t) \int_0^{r_1} dr \left(1 - kr^2\right)^{-1/2} .$$

If the universe is asymptotically flat with cosmological constant 0, then $k=\Lambda=0$ and the radial velocity is then

$$v_{rad} = \frac{\dot{R}(t_0)}{R(t_0)} D_{proper} \equiv H_0 D_{proper} .$$

Then for the parameters given,

$$D_{proper} = \frac{0.95 \times 300{,}000\ \text{km/s}}{15\times10^{-6}\ \text{km/sec/l-yr}} \approx 19\times10^9\ \text{l-yr} .$$

6. According to a "BELIEVE IT OR NOT!" cartoon (hyperlinked to this page)
"The amount of energy Americans use every day chewing bubble gum could light a city of 10 million inhabitants."

a) What is wrong with the way this problem is phrased?

Solution

Obviously, one wants to compare power, not energy.

b) Make a reasonable correction to the phrasing and determine whether the statement is true.

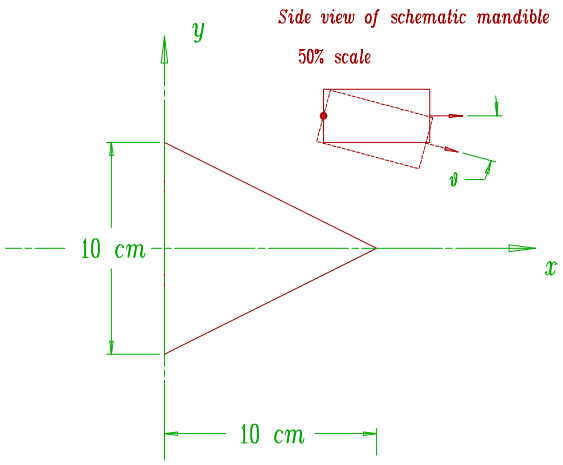
Solution

Suppose a city uses an average of 10^3 watts per inhabitant (probably an underestimate). Suppose also that an average of 50,000,000 Americans are chewing gum at any hour of the day or night (this is surely an overestimate). We must now estimate the work per chew and the power thus consumed. If model a human mandible (lower jaw) as a triangular object of base 10 cm, height 10 cm and thickness 5 cm, with unit specific gravity, its mass is 250 gm, and its cm must be raised about 2 cm at every chew, requiring work 0.05 J. The force to chew gum is surely much less than the maximum force a jaw can exert, which is several hundred pounds, say 1,000 Nt. So if we say the force is 100 Nt, exerted over a distance 0.02 m, we get 2 J, which is much greater than the work required to lift the jaw against gravity. Hence we shall neglect the latter. Now assume one chews once every two seconds (time yourself), the power consumed is one watt. (This is not a good way to lose weight!) The gum-chewers therefore emit an average power of 50,000 Kw. However, 10^7 Kw are needed to supply the city, hence Ripley's has misled us by at least a factor 200. The statement is FALSE.

c) Estimate the radiated power (in gravitational waves) from all this gum-chewing.

Solution

We need to estimate the quadrupole moment of a triangular object, hinged at one end, as shown:



The quadrupole moments are, taking the coordinates as shown,

$$Q^{xx} = \frac{325}{24} M$$

$$Q^{yy} = -\frac{125}{24} M$$

$$Q^{zz} = -\frac{50}{24} M$$

Now we rotate about the y-axis by an angle

$$\theta(t) = \theta_0 \cos(\omega t)$$

to obtain the time-dependent quadrupole moments. The time-dependent part is

$$Q^{jk}(t) = \left(\frac{Q^{xx} + Q^{yy}}{2}\right) \begin{pmatrix} \cos(2\theta) & 0 & \sin(2\theta) \\ 0 & 0 & 0 \\ -\sin(2\theta) & 0 & \cos(2\theta) \end{pmatrix}$$

giving the radiated power as (we make the small angle approximation, since θ_0 is small, replacing $\cos(2\theta)$ by 1 and $\sin(2\theta)$ by 2θ,

$$P = \frac{4G}{45c^5} \dddot{Q}^{rs} \dddot{Q}_{rs} \approx \frac{4G}{45c^5} \left(\frac{275\ \text{cm}^2}{24}\right)^2 (250\ \text{gm})^2\, \omega^6\, \theta_0^2$$

with a 15° opening angle and a frequency of 0.5 sec, we find $P \approx 1.3\times10^{-59}$ watts.

d) How does this compare with the electromagnetic radiation that would be emitted if the point of each gum-chewer's chin had one (1) unit of electron charge attached to it?

Solution

Here we have a charge moving (to a good approximation) linearly and sinusoidally, with amplitude

$$x_{\max} \approx 10\ \text{cm} \times \theta_0 = 2.6\ \text{cm}\ .$$

The dipole moment is then $p = -e\, x_{\max} \cos(\omega t)$ and the time-averaged power radiated is (see, *e.g.*, Jackson, *Classical Electrodynamics*)

$$P = \frac{\omega^4}{3c^3} e^2 x_{\max}^2 = 1.9\times10^{-55}\ \text{watts.}$$

A single electron radiates 10^4 more power in EM radiation than the jaw does gravitationally!