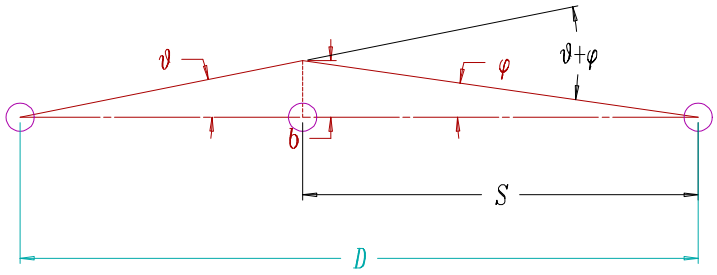
Solutions

1. A Cepheid variable star is 100 Kpsec away from the Earth, and a large unknown mass is directly in the line of sight, at an unknown distance from Earth. The gravitational lens effect (predicted by Einstein) says such a source will appear as a ring centered on the (invisible) occluding mass.

a) What is the angle subtended by the ring, as a function of the (unknown) mass of the intervening object and its (unknown) distance from Earth?



The problem involves small angle deflection of light. The total angle of deflection is

$$\theta + \varphi = \frac{b}{D-S} + \frac{b}{S} = \frac{4MG}{b}.$$

The angle subtended by the ring is

$$2\varphi = \frac{2b}{S} = \frac{2}{S}\left[\frac{4MG}{\frac{1}{D-S} + \frac{1}{S}}\right]^{1/2}$$

where we have used the first relation to solve for the impact parameter b.

b) Suppose the angular size of the ring is 3″ of arc. Express the unknown mass in solar masses, its unknown distance as a fraction of the Cepheid variable's distance from Earth, and find a relation between the (unknown) mass and its (unknown) distance.

Let $S = xD$, and $M = yM_\odot$;then

$$2\varphi = 2.77\times10^{-9}\ \text{rad} \times \left[y\left(\frac{1}{x} - 1\right)\right]^{1/2},$$

or

$$3'' = 5.7\times10^{-4}\ '' \times \left[y\left(\frac{1}{x} - 1\right)\right]^{1/2}$$

i.e.

$$y = 2.8\times10^{7} \cdot \frac{x}{1-x}$$

Thus an occluding object equal to the sun in mass would have to be closer than 4 light-days to produce such a large effect.

2. A photon of energy 5 eV collides head-on with an electron of energy 20 GeV ($1\ \text{GeV} = 10^9$ eV) and scatters elastically. What is the maximum energy of the scattered photon? At what angle (relative to its initial direction) is it scattered?

We use 4-momentum conservation:

$$\begin{pmatrix} k + p^0 \\ \vec{k} + \vec{p} \end{pmatrix} = \begin{pmatrix} k' + p'^0 \\ \vec{k}' + \vec{p}' \end{pmatrix}$$

where

$$\vec{p} = p\hat{z}$$

$$\vec{k} = -k\hat{z}$$

$$\vec{k}' \cdot \vec{p} = k' p \cos\theta \ .$$

Solve the momentum equation for $\vec{p}'$ and insert it in the energy equation, to get

$$\vec{p}' = \vec{p} + \vec{k} - \vec{k}'$$

$$m^2 + \left(\vec{p} + \vec{k} - \vec{k}'\right)^2 = m^2 + p^2 + 2\left(k - k'\right)p^0 + \left(k - k'\right)^2 \ .$$

Simplifying we find

$$\frac{1}{k'} = \frac{1}{k}\left(\frac{p^0 - p\cos\theta}{p^0 + p}\right) + \frac{1 + \cos\theta}{p^0 + p},$$

which is minimum (k' maximum) when $\cos\theta = 1$, *i.e.* when the photon is scattered 180° from its original direction. Since $p^0 = \sqrt{m^2 + p^2} \approx p + \frac{m^2}{2p}$, we find

$$k'_{\max} \approx \frac{p}{1 + \frac{m^2}{4pk}} = \frac{20\ \text{GeV}}{1 + (5\times10^{-4})^2/(4 \times 20 \times 5\times10^{-9})} = \frac{20\ \text{GeV}}{1 + .625} = 12.3\ \text{GeV}$$

3. In class we derived the Lienard-Weichart potential

$$A^{\mu}(x) = Q\,\frac{d\xi^{\mu}}{dt'}\left|\,|\vec{x}-\vec{\xi}(t')| - \frac{d\vec{\xi}}{dt'}\cdot\left(\vec{x}-\vec{\xi}(t')\right)\right|^{-1} \qquad (9.11)$$

for the vector potential of a moving charge, where

$$t' = t - |\vec{x}-\vec{\xi}(t')|$$

is the "retarded time". Use this expression to calculate the Poynting vector

$$\vec{S} = \frac{1}{4\pi}\vec{E}\times\vec{B}$$

far from an electric charge that is stopping in matter with constant deceleration, whose position is

$$\vec{\xi}(t) = \left(u\,t - \frac{1}{2}at^2\right)\hat{z}.$$

Calculate the total rate at which electromagnetic energy is radiated as a function of time.

(Hint: Use the relation $|\vec{x}-\vec{\xi}| \approx |\vec{x}| - \hat{x}\cdot\vec{\xi}$, $|\vec{x}| >> |\vec{\xi}|$, and don't forget that $\frac{d\xi^{\mu}}{dt'}$ depends on both position and time.)

We note that with the above approximation and the given expression for $\vec{\xi}(t)$ we may approximate t' by

$$t' \approx t - \frac{r}{c} + \frac{1}{c}\vec{\xi}(t')\cdot\hat{r} \approx \left(t - \frac{r}{c}\right)\left(1 - \frac{1}{c}\hat{r}\cdot\vec{u}\right)^{-1},$$

where we have written $|\vec{x}| = r, \hat{x} = \hat{r}$. Hence we may write

$$A^0 \approx \frac{Q}{r}\left(1 - \hat{r}\cdot\vec{\beta}\right)^{-1} + \mathcal{O}\left(r^{-2}\right)$$

and

$$\vec{A} \approx \frac{Q}{r}\,\vec{\beta}\left(1 - \hat{r}\cdot\vec{\beta}\right)^{-1} + \mathcal{O}\left(r^{-2}\right),$$

where

$$\vec{\beta} \overset{df}{=} \frac{1}{c}\hat{z}(u - at')$$

This gives, after a certain amount of manipulation,

$$\vec{E} = -\nabla A^0 - \frac{\partial\vec{A}}{\partial t} = \frac{-Qa}{rc^2}\left[\frac{\hat{r}\times(\hat{r}\times\hat{z}+\hat{z}\times\vec{\beta})}{(1-\hat{r}\cdot\vec{\beta})^3}\right] + O\left(\tfrac{1}{r^2}\right) = \frac{-Qa}{rc^2}\,\frac{\hat{r}\times(\hat{r}\times\hat{z})}{(1-\hat{r}\cdot\vec{\beta})^3}$$

and

$$\vec{B} = \nabla \times \vec{A} = \frac{Q\,a}{rc^2}\left[\frac{\hat{r}\times\hat{z}}{(1-\hat{r}\cdot\vec{\beta})^2} - \frac{(\hat{r}\cdot\hat{z})(\hat{r}\times\vec{\beta})}{(1-\hat{r}\cdot\vec{\beta})^3}\right]$$

For low velocities the Poynting vector reduces to

$$\vec{S} = \frac{c}{4\pi}\left(\frac{Qa}{rc^2}\right)^2 \sin^2\theta\ \hat{r}$$

giving the emitted power

$$P = \lim_{r\to\infty}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\, \vec{S}\cdot\hat{r} = \frac{2c}{3}\left(\frac{Qa}{c^2}\right)^2 .$$

4. Neutrinos (average energy 15 MeV) from the 1987A supernova were detected, within a time-frame ±3 hours, in coincidence with the light from the explosion. Since the supernova is about 150,000 light-years away, this coincidence permits us to set an upper bound on the *mass* of the electron neutrino. Estimate this upper bound.

The upper bound provided by this method is not very good:

$$\Delta t = 3\times 3600\ \text{sec} = 1.5\times10^5\ \text{yr}\times 3.17\times10^7\ \text{sec/yr}\times\left[\frac{1}{\beta}-1\right]$$

or

$$\frac{1}{\beta}-1 \approx \frac{m^2}{2p^2} = 2.3\times10^{-9},$$

from which we find $m \le 10^3\ \text{eV}/c^2$.

Also note that the photons and neutrinos passed through the *same* (spatially varying) gravitational potential supplied by the intervening matter. Estimate the maximum (gravitational) time-delay that would be caused by a mass equal to that of our galaxy (take its mass to be $10^{11}\,M_\odot$ and its radius to be 30,000 l-y), What does the fact that light and neutrinos arrived almost at the same time imply about neutrinos?

If we assume the galaxy is a uniform sphere of radius 30,000 l-y and that the Earth is on one side so the light and neutrinos came from an object 150,000 l-y on the opposite side, as shown, then the gravitational time delay will be

$$\Delta t = \int dz \left(\frac{1}{v(z)} - 1 \right) = -2 \int_{s-2R}^{R} dz \, \varphi(|z|)$$

where

$$\varphi(r) = \frac{-MG}{R} \begin{cases} \frac{3}{2} - \frac{r^2}{2R^2}, & r \le R \\ \frac{R}{r}, & r > R \end{cases} .$$

Plugging this in and doing the integral gives

$$\Delta t = \frac{GM}{Rc^3} \left[\ln\left(\frac{s-2R}{R} \right) + \frac{16}{3} \right]$$

which, with the numbers given, works out to 0.1 yr or 1.2 months.

Since the difference in time between the arrival of light, neutrinos and antineutrinos is much smaller than this (and in fact, consistent with 0) we may conclude that neutrinos, antineutrinos and photons behave esentially identically with respect to gravitation.

5. A massive scalar field theory is defined by the Lagrangian density

$$\mathscr{L} = \frac{1}{2} \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right) - \frac{\lambda}{4} \left(\varphi^2 - f^2 \right)^2$$

where m and f are real constants.

a) What is the Hamiltonian density of the field?

b) What is the total 3-momentum carried by the field? (Not the momentum-density!)

The Hamiltonian density is defined by

$$\mathscr{H} \overset{df}{=} \dot{\varphi} \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} - \mathscr{L} = \frac{1}{2} \left(\dot{\varphi} \dot{\varphi} + \nabla\varphi \cdot \nabla\varphi + m^2 \varphi^2 \right) + \frac{\lambda}{4} \left(\varphi^2 - f^2 \right)^2$$

and the obvious generalization to the energy momentum tensor density is

$$\mathscr{T}^{\mu\nu} \overset{df}{=} \frac{1}{2} \partial^\mu \varphi \frac{\partial \mathscr{L}}{\partial \varphi_{,\nu}} + \frac{1}{2} \partial^\nu \varphi \frac{\partial \mathscr{L}}{\partial \varphi_{,\mu}} - \eta^{\mu\nu} \mathscr{L} .$$

Thus the 3-momentum density is

$$\mathscr{T}^{0k} = \partial^0 \varphi \, \partial^k \varphi - \eta^{0k} \mathscr{L} \equiv -\dot{\varphi} \nabla_k \varphi$$

hence the total 3-momentum of the field is just

$$\vec{P} = -\int d^3x \, \dot{\varphi}(x) \nabla \varphi(x) .$$