We will be using complex variables to describe waves throughout the course, so it will be important for all students to be comfortable manipulating them. This handout gives a few examples of how to do complex math.

**Simple Operations**

Suppose $z_1 = 2 + 3i$ and $z_2 = -2 + 2i$. Then

$$z_1 + z_2 = (2 + -2) + (3 + 2)i = 5i$$

and

$$z_1 z_2 = 2 \times (-2) + 2 \times (2i) + (3i) \times (-2) + (3i) \times (2i) = -10 - 2i.$$

Multiplying by a real number is particularly easy: $3z_1 = 3 \times 2 + 3 \times (3i) = 6 + 9i$.

Division is a bit harder. A good method is to write $z_1/z_2 = (z_1 z_2^*)/(z_2 z_2^*)$, since the product $z_2 z_2^* \equiv |z_2|^2$ is always real. This reduces complex division to a complex multiplication and a real division. So,

$$\frac{z_1}{z_2} = \frac{(2+3i)(-2-2i)}{(-2+2i)(-2-2i)} = \frac{2-10i}{8} = 0.25 - 1.25i$$

**Polar Form**

The form $x + iy$ is sometimes called Cartesian form. It is often convenient to instead express complex variables in polar form, $z = re^{i\theta}$. Here $r$ is the magnitude and $\theta$ is the phase. To go from Cartesian to polar form, use

$$r = |z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

and

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \sin^{-1}\left(\frac{y}{r}\right) = \cos^{-1}\left(\frac{x}{r}\right).$$

Thus we have $z_1 = 3.61 \exp(i0.983)$, where the angle is in radians. This can also be expressed as $z_1 = 3.61\angle 0.983$ rad or $3.61\angle 56.3°$. We will mostly be working with variables rather than actual numbers, however, and the $re^{i\theta}$ form will be most convenient.

An advantage of polar form is that multiplication and division are easy: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)},$$

following the usual rules for multiplying exponentials. So, another way to do division is to convert the numbers to polar form and then just divide the magnitudes and subtract the phases. Just to check, we have $z_1 = 3.61\angle 56.3°$ and $z_2 = 2.83\angle 135°$, so $z_1/z_2 = 3.61/2.82\angle(56.3-135) = 1.27\angle -78.7°$ which in Cartesian form is $0.25-1.25i$, as we obtained above.

To go from polar form to Cartesian form, use

$$x = r\cos\theta$$

and

$$y = r\sin\theta.$$

So, $4\angle 60° = 2 + 3.46i$.

**Calculating the Magnitude**

It is often necessary to calculate the magnitude of a complicated expression. For instance, we might want to know

$$A = \left|\frac{(p+iq)e^{-\alpha+ikd}}{1+ikd}\right|.$$

It would be a big mistake to try to work out the the real and imaginary parts of this expression to apply $|z| = \sqrt{x^2+y^2}$. Instead, use two simple techniques. First, the magnitude of a product is equal to the product of the magnitudes: $|z_1 z_2| = |z_1||z_2|$. The same holds for division. So in our example, we have

$$A = \frac{|p+iq| \times |e^{-\alpha+ikd}|}{|1+ikd|} = \frac{\sqrt{p^2+q^2}}{\sqrt{1+k^2d^2}}|e^{-\alpha+ikd}|.$$

We could use the same rule on the exponential if we rewrote it as a product of $e^{-\alpha}$ and $e^{ikd}$. Or we can use the second technique, which starts with $|z| = \sqrt{zz^*}$. To obtain $z^*$ we simply replace all the $i$'s in $z$ by $-i$. In our example, we have

$$|e^{-\alpha+ikd}| = \left(e^{-\alpha+ikd}e^{-\alpha-ikd}\right)^{1/2} = e^{-\alpha}$$

since $e^{ikd}e^{-ikd} = 1$. So we get the final result

$$A = \left(\frac{p^2+q^2}{1+k^2d^2}\right)^{1/2} e^{-\alpha}.$$

The ease of calculating $z^*$ can also be used to find the real and imaginary parts of an expression, since

$$\text{Re } z = \frac{1}{2}(z+z^*)$$

and

$$\text{Im } z = \frac{1}{2i}(z-z^*).$$

This comes in handy sometimes.

**Calculus**

Integrals and derivatives don't pay any attention to whether a function is real, imaginary, or complex. So for instance,

$$\frac{d}{du}\left(\frac{1}{1+iu}\right) = \frac{-i}{(1+iu)^2}$$

and

$$\int_{-\pi/2}^{\pi/2} e^{i\theta} d\theta = \frac{1}{i} e^{i\theta}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{i}\left(e^{i\pi/2} - e^{-i\pi/2}\right) = 2$$

where the final result is obtained using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$. Integrals get more interesting when the variable being integrated over is complex, but we won't be doing much of that in this class.

**Final Example**

This is a homework problem from a previous year:

> Suppose a harmonic wave $\psi(x,t)$ is expressed as $\psi = \text{Re } Ae^{i(kx-\omega t)}$, where $A$ may be complex. Find an expression for the time average $\langle\psi^2\rangle$, where
>
> $$\langle f\rangle \equiv \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt.$$

Let's solve this. We need to calculate

$$\int_{-T/2}^{T/2} \psi(x,t)^2 dt.$$

To get an explicit expression for $\psi$, take $A = |A|e^{i\phi}$, so that $\psi = \text{Re } |A|e^{i(kx-\omega t+\phi)} = |A|\cos(kx - \omega t + \phi)$. Then we need

$$\int_{-T/2}^{T/2} |A|^2 \cos^2(kx - \omega t + \phi)dt = |A|^2 \int_{-T/2}^{T/2} \frac{1 + \cos 2(kx - \omega t + \phi)}{2} dt,$$

where I used a trig identity to obtain the right-hand side. The integral is then

$$\frac{|A|^2}{2}\left[T - \frac{\sin(2kx - \omega T + 2\phi)}{2\omega} + \frac{\sin(2kx + \omega T + 2\phi)}{2\omega}\right].$$

Dividing by $T$ and taking $T \to \infty$ yields the result

$$\langle\psi^2\rangle = \frac{|A|^2}{2}.$$

Here we hardly needed any complex math at all, just enough to understand how to get an explicit expression for $\psi$ from the information given.