In this handout, I will go through the derivations of some of the results I gave in class (Lecture 14, 10/21). I won't reintroduce the concepts though, so if you haven't seen the lecture, you should watch it first. I'll try to provide enough detail in the calculations to illustrate some of the techniques for working with Fourier transforms. I hope it is helpful, but you shouldn't need any of this specific material for the homework assignments or final exam.

1 Delta Functions

I defined

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,dt \tag{1}$$

and explained why $\delta(\omega) = 0$ for $\omega \neq 0$ but $\delta(0) = \infty$. A crucial property of the delta function, however, is that

$$\int_{-\infty}^{\infty} \delta(\omega)\,d\omega = 1.$$

I'll derive this property here.

The infinite bounds on the integrals cause some difficulty, so let us interpret

$$\int_{-\infty}^{\infty} e^{i\omega t}\,dt$$

as

$$\lim_{T\to\infty}\int_{-T}^{T} e^{i\omega t}\,dt.$$

Then define

$$\delta_T(\omega) = \frac{1}{2\pi}\int_{-T}^{T} e^{i\omega t}\,dt$$

so that $\delta(\omega) = \lim_{T\to\infty}\delta_T(\omega)$. We can simply integrate to get δ_T:

$$\begin{aligned}
\delta_T(\omega) &= \frac{1}{2\pi}\int_{-T}^{T} e^{i\omega t}\,dt \\
&= \frac{1}{2\pi i\omega} e^{i\omega t}\Big|_{-T}^{T} \\
&= \frac{1}{2\pi i\omega}\left(e^{i\omega T} - e^{-i\omega T}\right) \\
&= \frac{1}{\pi\omega}\sin(\omega T)
\end{aligned}$$

Now we need to evaluate $\int_{-\infty}^{\infty}\delta_T(\omega)\,d\omega$. This is a bit harder. The professional way to do it is using contour integration, but we can avoid that with some simple tricks. First, note that $\sin(\omega)/\omega$ is symmetric, so

$$\int_{-\infty}^{\infty}\delta_T(\omega)\,d\omega = \frac{2}{\pi}\int_0^{\infty}\frac{\sin(\omega T)}{\omega}\,d\omega.$$

Change variables to $u = \omega T$, giving

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin(u)}{u}\, du.$$

Now, make the unintuitive substitution

$$\frac{1}{u} = \int_0^{\infty} e^{-uv}\, dv$$

to get

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta_T(\omega)\, d\omega &= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-uv} \sin(u)\, dv\, du \\
&= \frac{1}{i\pi} \int_0^{\infty} \int_0^{\infty} e^{-uv} \left(e^{iu} - e^{-iu}\right)\, du\, dv \\
&= \frac{1}{i\pi} \int_0^{\infty} \int_0^{\infty} \left[e^{u(-v+i)} - e^{u(-v-i)}\right]\, du\, dv
\end{aligned}$$

The u integrals are easily done, giving

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta_T(\omega)\, d\omega &= \frac{1}{i\pi} \int_0^{\infty} \left(\frac{-1}{-v+i} + \frac{1}{-v-i}\right)\, dv \\
&= \frac{1}{i\pi} \int_0^{\infty} \frac{2i}{1+v^2}\, dv \\
&= \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+v^2}\, dv
\end{aligned}$$

The v integral is again elementary. Substitute $v = \tan\theta$, so that $dv \to \sec^2\theta\, d\theta$ and the limits become 0 to $\pi/2$. But also, $1+v^2$ becomes $1+\tan^2\theta = \sec^2\theta$ which cancels the $\sec^2\theta$ term from the differential. The integral becomes $\int_0^{\pi/2} d\theta = \pi/2$, so that

$$\int_{-\infty}^{\infty} \delta_T(\omega)\, d\omega = 1.$$

Since this holds independently of T, it is reasonable to conclude that

$$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1. \tag{2}$$

There are some formal mathematical issues regarding the limit $T \to \infty$, but we don't need to worry about them here.

Our result (2), combined with the other properties of $\delta(\omega)$ that we know, is enough to establish that for any function $F(\omega)$,

$$\int F(\omega)\delta(\omega - \omega_0)\, d\omega = F(\omega_0) \tag{3}$$

which is the main result we'll need.

2 Calculating Transforms

On slide 29, I gave a table with the Fourier transforms of several functions. We did the first in class, and I'll go through the rest here.

The second and third lines of the table are easy. If $f(t) = e^{-i\omega_0 t}$ then

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} e^{-i\omega_0 t}\, dt \\ &= \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t}\, dt \end{aligned}$$

By definition (1), this is

$$F(\omega) = 2\pi\delta(\omega - \omega_0)$$

as the table gives. Note that by choosing $\omega_0 = 0$, this gives us the transform of $f(t) = 1$.

If $f(t) = \delta(t - t_0)$, then

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} \delta(t - t_0)\, dt \\ &= e^{i\omega t_0}. \end{aligned}$$

Note that $\delta(t)$ works just like $\delta(\omega)$, or any other variable you might need to use. You could more generally write (3) as

$$\int_{-\infty}^{\infty} F(u)\delta(u - u_0)\, du = F(u_0)$$

for any variable u.

The fourth transform listed takes more work. Say $f(t) = e^{-t^2/\tau^2}$. Then

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega t} dt \\ &= \int_{-\infty}^{\infty} e^{-t^2/\tau^2 + i\omega t} dt. \end{aligned}$$

We can simplify the integrand with a trick called "completing the square." Note that

$$-\frac{t^2}{\tau^2} + i\omega t = -\frac{t^2}{\tau^2} + i\omega t - \left(\frac{i\omega\tau}{2}\right)^2 + \left(\frac{i\omega\tau}{2}\right)^2,$$

since we're just adding and subtracting the same thing. But the first three terms can be factored:

$$\left[-\frac{t^2}{\tau^2} + i\omega t - \left(\frac{i\omega\tau}{2}\right)^2\right] + \left(\frac{i\omega\tau}{2}\right)^2 = \left[-\left(\frac{t}{\tau} - \frac{i\omega\tau}{2}\right)^2\right] - \frac{\omega^2\tau^2}{4}.$$

So we get

$$F(\omega) = e^{-\omega^2\tau^2/4} \int_{-\infty}^{\infty} e^{-(t/\tau - i\omega\tau/2)^2}\, dt$$

Change variables to $u = t/\tau - i\omega\tau/2$. Then $dt \to \tau\, du$, and the limits of the integral don't change since they are infinite. Actually, that's a bit tricky: the limits really become something like $\infty - i\omega\tau/2$, which isn't quite the same thing as plain ∞. Sometimes the imaginary bit matters, but not here, so we'll just ignore it. (For people who know something about contour integrals, we can displace the integration path back to the real axis because the integrand has no poles.)

In any case, we get

$$F(\omega) = \tau e^{-\omega^2\tau^2/4} \int_{-\infty}^{\infty} e^{-u^2}\, du.$$

You could just look the final integral up, but I'll show you how to solve it for yourself. Define its value to be I:

$$I = \int_{-\infty}^{\infty} e^{-u^2}\, du.$$

Then consider

$$I^2 = \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv$$

where we introduce v to keep track of which integral is which. Combined, we have

$$I^2 = \iint e^{-(u^2+v^2)}\, du\, dv.$$

Now do this double integral in polar coordinates: $(u, v) \to (r, \theta)$, where $u = r\cos\theta$ and $v = r\sin\theta$. Then $du\, dv \to r\, dr\, d\theta$ and we have

$$I^2 = \int_0^{2\pi} \int_0^{\infty} re^{-r^2}\, dr\, d\theta$$

The θ integral gives a factor of 2π, and the r integral can be done with the substitution $s = r^2$. Then $ds = 2r\, dr$ so

$$I^2 = \pi \int_0^{\infty} e^{-s}\, ds.$$

The s integral is just unity, so $I^2 = \pi$. But then $I = \sqrt{\pi}$ and

$$F(\omega) = \tau\sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$

as the table indicates.

3 Convolution Theorem

The convolution theoerm states that if $F(\omega) = F_1(\omega)F_2(\omega)$, then

$$f(t) = \int_{-\infty}^{\infty} f_1(T) f_2(t - T)\, dT,$$

where F, F_1 and F_2 are respectively the transforms of f, f_1 and f_2. This is fairly easy to prove.

We have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} \, d\omega$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) F_2(\omega) e^{-i\omega t} \, d\omega.$$

But in turn,

$$F_1(\omega) = \int_{-\infty}^{\infty} f_1(t_1) e^{i\omega t_1} \, dt_1$$

and

$$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t_2) e^{i\omega t_2} \, dt_2.$$

Again, we can always relabel integration variables as we like.

Combining all these, we have

$$f(t) = \frac{1}{2\pi} \iiint f_1(t_1) f_2(t_2) e^{i\omega(t_1+t_2-t)} \, dt_1 \, dt_2 \, d\omega.$$

Do the ω integral first, so that

$$f(t) = \frac{1}{2\pi} \iint f_1(t_1) f_2(t_2) \left[\int_{-\infty}^{\infty} e^{i\omega(t_1+t_2-t)} \, d\omega \right] dt_1 \, dt_2.$$

You should recognize the expression in brackets as $2\pi\delta(t_1 + t_2 - t)$, so

$$f(t) = \iint f_1(t_1) f_2(t_2) \delta(t_1 + t_2 - t) \, dt_1 \, dt_2.$$

Use the δ-function to do the t_2 integral, so that we can replace t_2 with $t - t_1$ to get

$$f(t) = \int_{-\infty}^{\infty} f_1(t_1) f_2(t - t_1) \, dt_1.$$

Relabling $t_1 \to T$ gives us our convolution result.

The correlation theorem and Parseval's theorem have dervations very similar to that of the convolution theorem. I'll go through Parseval's to demonstrate. The theorem states that

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

if f and F are a Fourier transform pair. Consider

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t_1) e^{i\omega t_1} \, dt_1 \right] \left[\int_{-\infty}^{\infty} f^*(t_2) e^{-i\omega t_2} \, dt_2 \right] d\omega.$$

Again, reorder the integral to do the ω one first:

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \iint f(t_1) f^*(t_2) \left[\int_{-\infty}^{\infty} e^{i\omega(t_1 - t_2)} \, d\omega \right] dt_1 \, dt_2.$$

The term in brackets is $2\pi\delta(t_1 - t_2)$ so

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1) f^*(t_2) \delta(t_1 - t_2) \, dt_1 \, dt_2$$

and using the δ-function gives Parseval's theorem:

$$\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} f(t_1) f^*(t_1) \, dt_1 = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

4 Transform pairs

Finally, one thing I didn't really go through in class is how to get the inverse Fourier transform of a function $F(\omega)$ that has the same "form" as some $g(t)$ whose transform you know already. For instance, in class (slide 24), I calculated the inverse transform of a square pulse function $F(\omega)$. We already knew the transform of a square pulse in time, and we could have used that information to obtain $f(t)$ rather than doing the integral. The technique is nice to know, so I'll go through it here.

Say we know that $g(t)$ has transform $G(\omega)$. Then suppose we're given $F(\omega) = g(\omega)$. That looks a little funny, so here's an example: Say $g(t) = \cos(at)$ for some constant a. Then $G(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)]$. The question is: if we're given a transform $F(\omega) = \cos(a\omega)$, what is $f(t)$? Obviously, a must have different units in F, but the form is the same as g.

We want

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \, d\omega. \end{aligned}$$

Now we know

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} \, d\omega$$

so just relabeling variables $t \to \omega$ and $\omega \to t_1$ gives

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t_1) e^{-i\omega t_1} \, dt_1.$$

Substitute this into the expression for $f(t)$:

$$f(t) = \frac{1}{(2\pi)^2} \iint G(t_1) e^{-i\omega(t + t_1)} \, dt_1 \, d\omega.$$

Once again, do the ω integral first

$$\begin{aligned} f(t) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} G(t_1) \left[\int_{-\infty}^{\infty} e^{-i\omega(t+t_1)}\, d\omega \right] dt_1 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(t_1) \delta(t+t_1)\, dt_1 \\ &= \frac{1}{2\pi} G(-t) \end{aligned}$$

which is the desired result.

So in our example, if we have $F(\omega) = \cos(a\omega)$, then

$$f(t) = \frac{1}{2\pi} \left[\delta(-t-a) + \delta(-t+a) \right].$$

This can be simplified since $\delta(-t) = \delta(t)$, so

$$f(t) = \frac{1}{2\pi} \left[\delta(t+a) + \delta(t-a) \right].$$

For another example, apply this to the calulation we did in class. We wanted the inverse transform of

$$F(\omega) = \begin{cases} 1 & \text{if } |\omega| < \omega_m \\ 0 & \text{else} \end{cases}$$

We know that the function g_1

$$g_1(t) = \begin{cases} \frac{1}{\tau} & \text{if } |t| < \frac{\tau}{2} \\ 0 & \text{else} \end{cases}$$

has Fourier transform

$$G_1(\omega) = \text{sinc}\left(\frac{\omega\tau}{2} \right).$$

Use the linearity properties to see that

$$g(t) = \begin{cases} 1 & \text{if } |t| < a \\ 0 & \text{else} \end{cases}$$

has transform $G(\omega) = 2a \,\text{sinc}(\omega a)$. Then our $F(\omega) = g(\omega)$ with $a = \omega_m$, so

$$f(t) = \frac{1}{2\pi} G(-t) = \frac{1}{2\pi} \left[2\omega_m \,\text{sinc}(-t\omega_m) \right] = \frac{\omega_m}{\pi} \,\text{sinc}(\omega_m t)$$

since $\text{sinc}(-\omega t) = \text{sinc}(\omega t)$. This agrees with the result we got in class by just doing the integral.

5 Summary

I've presented a pretty quick run through some of the important properties of Fourier transforms. Again, a lot of this is more complicated than what we'll be doing in class: I'll try to focus more on optics than on math. Nonetheless, you will be seeing many of these ideas again, and I hope that filling out some of the steps will help everyone feel more confident using the Fourier technique.