Fourier Approach to Wave Propagation

Last time, reviewed Fourier transform

Write any function of space/time =
sum of harmonic functions $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

Actual waves:
harmonic functions restricted $k^2 = n^2\omega^2/c^2$

Today, apply Fourier to wave propagtion

Start to study diffraction

Outline:

- Diffraction
- Fourier approach
- Transfer function
- Fresnel approximation
- Gaussian example

Note: we won't be following book very well
- Hecht Ch. 10 takes different approach
- Ch. 11: Fourier approach, based on Ch. 10

Next time, continue development

Diffraction

Previously said ray optics fails

- small feature sizes a
- long propagation distances d

Need $d \ll a^2/\lambda$

Otherwise see *diffraction*

light spreads out

Demo!

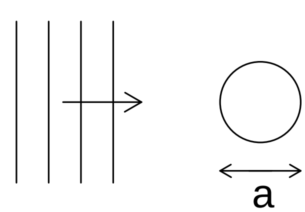
Want to understand diffraction

and calculate effects

Note: already have one way to understand:

scattering picture

Recall HW 2:



Plane wave incident on sphere

diameter a

Ray optics:

Transmitted light has shadow diameter a

Propagates indefinitely

Wrong!

Scattering picture:

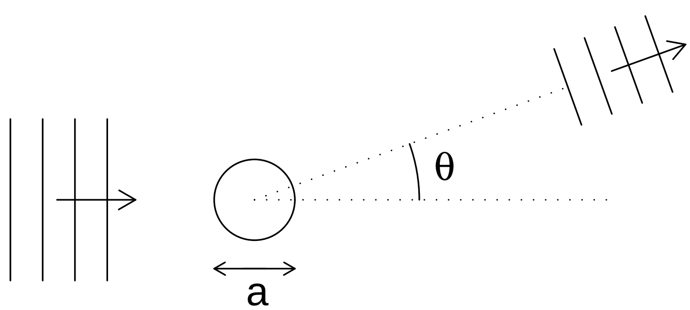
Shadow due to forward scattered field

In shadow, $E_{\text{tot}} = E_{\text{inc}} + E_{\text{scat}} \approx 0$

To sides, E_{scat} fields cancel out

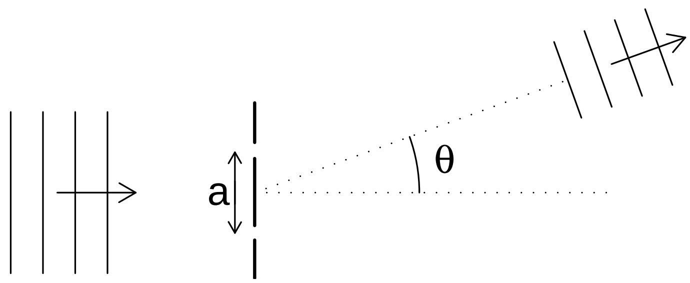
But forward scattering not perfectly forward

at angle $\theta \sim \lambda/a$, E_{scat} significant



At small angle, E_{scat} from all atoms $\approx$ in phase

Similar to two slit interference



Get large peak when fields from slits in phase

Diffraction in scattering picture:
E_{scat} fields don't cancel perfectly for finite object

General prediction:
    Diffraction angle $\theta \approx \lambda/a$

Valid, but hard to calculate more precisely
    Come back to idea later

Fourier Treatment

Use math

Set up problem:

Suppose monochromatic field, frequency ω
propagating towards $+z$ (perhaps at angle)

Specify $E(\mathbf{r}, t)$ in plane $z = 0$
(= plane of slits, aperture)
Ask: What is $E(\mathbf{r}, t)$ for $z > 0$?

Don't worry about 3D objects like sphere
Sphere $\approx$ disk

Monochromatic: write $E(\mathbf{r}, t) = E(\mathbf{r})e^{-i\omega t}$
just consider $E(\mathbf{r})$

Field known at $z = 0$:
Write $E(x, y, z = 0) = A(x, y)$

Call $A(x, y)$ = *aperture function*

Usually look at diffraction from aperture
$A(x, y) = 0$ for points outside aperture
$A(x, y) = E(x, y, 0)$ for points inside aperture

(Stop using A for amplitude)

Example:

Plane wave $E_{\text{inc}} = E_0 e^{i[k(z\cos\theta + x\sin\theta) - \omega t]}$
travelling at angle θ to z-axis

Incident on square aperture side a,
centered at $x = x_0$, $y = y_0$

Then

$$A(x,y) = \begin{cases} E_0 e^{ikx\sin\theta} & (|x - x_0|, |y - y_0| < a/2) \\ 0 & \text{else} \end{cases}$$

Think of $A(x,y)$ as initial condition
want to solve for $E(x,y,z)$

Apply Fourier ideas

First thought:

$$E(x,y,z) = \frac{1}{(2\pi)^3} \iiint \mathcal{E}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3k$$

If we knew $\mathcal{E}(\mathbf{k})$, problem solved

Do have

$$A(x,y) = \frac{1}{(2\pi)^3} \iiint \mathcal{E}(\mathbf{k}) e^{i(k_x x + k_y y)}\, d^3k$$

Can we invert to get $\mathcal{E}(\mathbf{k})$ from $A(x,y)$?

No: $\mathcal{E}(\mathbf{k}) = \iiint E(x,y,z) e^{i(k_x x + k_y y + k_z z)}\, dx\, dy\, dz$

Second thought:

Have $A(x,y) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x,k_y) e^{i(k_x x + k_y y)}\, dk_x\, dk_y$

with $\mathcal{A}(k_x,k_y) = \iint A(x,y) e^{i(k_x x + k_y y)}\, dx\, dy$

No problem getting $\mathcal{A}(k_x,k_y)$

Can we get $E(x,y,z)$ from $\mathcal{A}$?
Yes!

Develop with example:

Suppose $A(x,y) = E_0 e^{i(\beta x)}$
harmonic function

What function $E(x,y,z)$ would give us this A?

Already know answer:

$$E(\mathbf{r}) = E_0 e^{i(\beta x + k_z z)} \text{ for some } k_z$$

Plane wave

In this case, easy to guess form of solution

What is k_z?

Have $k^2 = k_x^2 + k_y^2 + k_z^2 = n^2\omega^2/c^2$
ω, n given

For our function $k_x = \beta$ and $k_y = 0$, so

$$k_z^2 = k^2 - \beta^2$$

$$k_z = \sqrt{k^2 - \beta^2}$$

Full solution is

$$E(\mathbf{r}) = E_0 e^{i(\beta x + z\sqrt{k^2-\beta^2})}$$

Question: Could I use $k_z = -\sqrt{k^2 - \beta^2}$ instead?

In general, if $A(x, y) = E_0 e^{i(k_x x + k_y y)}$, get solution

$$E(\mathbf{r}) = E_0 e^{i(k_x x + k_y y + \kappa z)}$$

for $\kappa \equiv \sqrt{k^2 - k_x^2 - k_y^2}$

Solution to problem for particular form $A(x, y)$

Important to understand this!

Question: If $A(x, y) = E_0$, what is $E(x, y, z)$?

With Fourier transform,

$A(x,y) =$ sum of harmonic funcs

So solution $E(\mathbf{r}) =$ sum of plane waves
with $\kappa = \sqrt{k^2 - k_x^2 - k_y^2}$

If $A(x,y) = \dfrac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y)}\, dk_x\, dk_y$

then $\boxed{E(\mathbf{r}) = \dfrac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y + \kappa z)}\, dk_x\, dk_y}$

Simple example: $A(x,y) = E_0 \cos(\beta x)$

One solution:

$$A(x,y) = \frac{E_0}{2}\left(e^{i\beta x} + e^{-i\beta x}\right)$$
$$= \text{ sum of harmonic funcs}$$

Then

$$E(\mathbf{r}) = \frac{E_0}{2}\left(e^{i(\beta x + z\sqrt{k^2-\beta^2})} + e^{i(-\beta x + z\sqrt{k^2-\beta^2}}\right)$$
$$= E_0 e^{iz\sqrt{k^2-\beta^2}} \cos(\beta x)$$

Another solution:

Recall transform of $e^{i\beta x}$ is $2\pi\delta(k_x - \beta)$

So

$$\mathcal{A}(k_x, k_y) = 2\pi^2 E_0 \Big[\delta(k_x - \beta) + \delta(k_x + \beta)\Big]\delta(k_y)$$

So

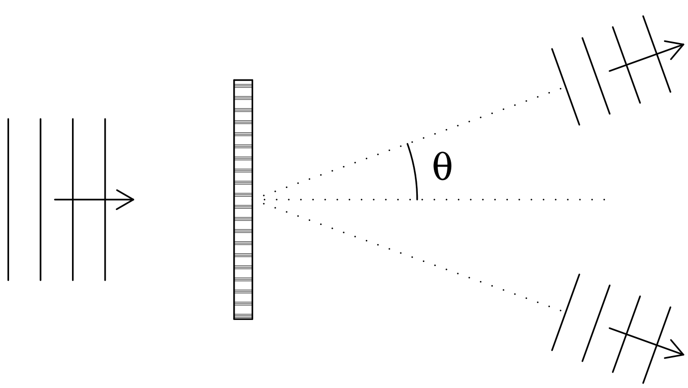
$$\begin{aligned} E(\mathbf{r}) &= \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y + \kappa z)}\, dk_x\, dk_y \\ &= \frac{1}{(2\pi)^2}\Big\{2\pi^2 E_0 \Big[e^{i(\beta x + \kappa z)} + e^{i(-\beta x - \kappa z)}\Big]\Big\} \\ &= E_0 e^{i\kappa z} \cos(\beta x) \end{aligned}$$

for $\kappa = \sqrt{k^2 - \beta^2}$

Either method fine

Note, solution is physically interesting:

Two plane waves, angle $\theta = \tan^{-1}(\beta/\kappa)$



Implement with glass plate, sinusoidal markings

simple diffraction grating

Another example:

Plane wave normally incident on square hole

$$A(x,y) = \begin{cases} 1 & (|x|, |y| < a/2) \\ 0 & (\text{else}) \end{cases}$$

Then

$$\begin{aligned} \mathcal{A}(k_x, k_y) &= \iint A(x,y) e^{i(k_x x + k_y y)}\, dx\, dy \\ &= \left(\int_{-a/2}^{a/2} e^{ik_x x}\, dx \right) \left(\int_{-a/2}^{a/2} e^{ik_y y}\, dy \right) \\ &= a^2 \,\text{sinc}\left(\frac{k_x a}{2}\right) \text{sinc}\left(\frac{k_y a}{2}\right) \end{aligned}$$

and

$$E(\mathbf{r}) = \frac{a^2}{(2\pi)^2} \iint \text{sinc}\left(\frac{k_x a}{2}\right) \text{sinc}\left(\frac{k_y a}{2}\right) \times e^{i\left(k_x x + k_y y + z\sqrt{k^2 - k_x^2 - k_y^2}\right)}\, dk_x\, dk_y$$

Can't do this integral analytically

- square root in exponent is hard!

Need to introduce some approximations

First, study what we'll approximate

Transfer Function

General result

$$E(\mathbf{r}) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y + \kappa z)}\, dk_x\, dk_y$$

Can write as

$$E(\mathbf{r}) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y)\mathcal{H}(k_x, k_y) e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

for $\mathcal{H}(k_x, k_y) = e^{iz\sqrt{k^2 - k_x^2 - k_y^2}}$

Call $\mathcal{H}$ = *transfer function* for free space

Note $\mathcal{H}$ depends on z = propagation distance

More general:

$$\boxed{\mathcal{H}_d(k_x, k_y) = e^{id\sqrt{k^2 - k_x^2 - k_y^2}}}$$

propagates field from z_0 to $z_0 + d$

Call $E(x, y, z_0)$ = input, $E(x, y, z_0 + d)$ = ouput

Linear system: output depends linearly on input

Transfer function = linear coefficients
but in Fourier space

$\mathcal{H}_d = e^{id\sqrt{k^2-k_x^2+k_y^2}}$

For $k_x^2 + k_y^2 < k^2$, have $|\mathcal{H}| = 1$

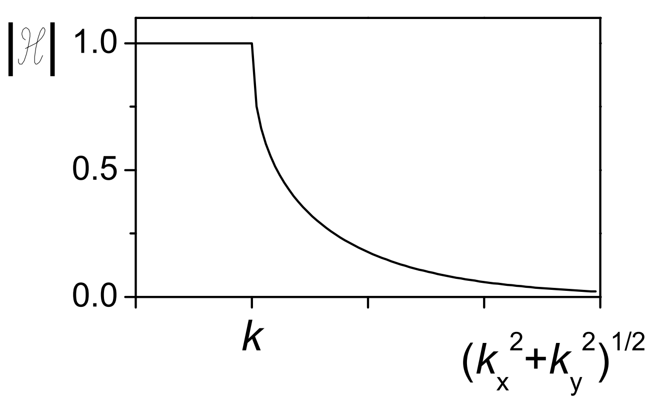
k_z is real

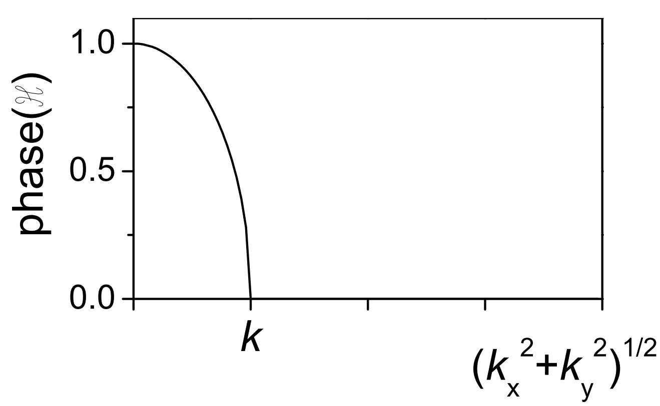
But for $k_x^2 + k_y^2 > k^2$, have

$$|\mathcal{H}| = e^{-d\sqrt{k_x^2+k_y^2-k^2}} < 1$$

k_z is imaginary!

Plot magnitude and phase





Is it possible to have $k_x^2 + k_y^2 > k^2$?

Yes: can make arbitrary apertures

If feature size $\lesssim \lambda$, will have
$\mathcal{A}(k_x, k_y) \neq 0$ for large k_x, k_y

Example: square hole with $a = 10$ nm

For large k_x, k_y, $\mathcal{H}$ decays with d
$\Rightarrow E(\mathbf{r})$ decays with d

Have seen before: evanescent wave

For aperture with small hole,
field doesn't propagate away

Can't "fit" wave through hole smaller than $\lambda/2\pi$

Limits imaging resolution of microscope:
images of small features don't propagate

But, can measure evanescent wave itself:
called *near field microscopy*

Place detector very close to surface
resolution $\approx$ surface distance/2π

Fresnel Approximation

Note, large $k_x, k_y \Rightarrow$ large propagation angle θ

$$\sin\theta = \frac{\sqrt{k_x^2 + k_y^2}}{k}$$

But usually interested in small $\theta \approx$ paraxial
Evanescent wave behavior irrelevant

Suggests approximation

$$\sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{k_x^2 + k_y^2}{2k}$$

so $\mathcal{H}_d \approx e^{ikd}\, e^{-id(k_x^2+k_y^2)/2k}$

Called Fresnel approximation

Gives diffracted field $E(x, y, z) =$

$$\frac{e^{ikz}}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y)}\, e^{-iz(k_x^2+k_y^2)/2k}\, dk_x\, dk_y$$

Integrals more manageable

Still hard to get analytic result
but numerical integration is straightforward

Valid when next term in expansion is small

Next term in Taylor series of $d\sqrt{k^2 - k_x^2 - k_y^2}$

$$= \frac{d(k_x^2 + k_y^2)}{8k^3}$$

For propagation angle

$$\theta \approx \frac{\sqrt{k_x^2 + k_y^2}}{k}, \text{ need } kd\theta^4 \ll 1$$

More physics of Fresnel approxmation next class

For now:
one example where analytic solution possible

Gaussian beam

Suppose $A(x, y) = E_0 e^{-(x^2+y^2)/w_0^2}$

Gaussian function, width w_0

Make with glass filter:
- transparent in center
- smoothly becomes opaque at edge

Turns out, this field produced naturally by laser
$\rightarrow$ practically important

Calculate $E(x, y, z)$

Need transform $\mathcal{A}(k_x, k_y)$

Transform of Gaussian e^{-x^2/w_0^2} is $w_0\sqrt{\pi}e^{-w_0^2 k_x^2/4}$

So $\mathcal{A}(k_x, k_y) = E_0 \pi w_0^2 e^{-w_0^2(k_x^2+k_y^2)/4}$

With Fresnel approximation

$$E(\mathbf{r}) = \frac{e^{ikz}}{(2\pi)^2} E_0 \pi w_0^2$$

$$\times \iint e^{-w_0^2(k_x^2+k_y^2)/4}\, e^{-iz(k_x^2+k_y^2)/2k}\, e^{i(k_x x+k_y y)}\, dk_x\, dk_y$$

Define $q^2 = w_0^2 + i2z/k$

Then

$$E(\mathbf{r}) = \frac{e^{ikz}}{(2\pi)^2} E_0 \pi w_0^2$$

$$\times \iint e^{-q^2(k_x^2+k_y^2)/4}\, e^{i(k_x x+k_y y)}\, dk_x\, dk_y$$

$$= e^{ikz} E_0 \pi w_0^2$$
$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-q^2 k_x^2/4}\, e^{ik_x x}\, dk_x$$
$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-q^2 k_y^2/4}\, e^{ik_y y}\, dk_y$$

Inverse transforms of Gaussians

So

$$E(\mathbf{r}) = E_0 e^{ikz} \pi w_0^2 \left(\frac{1}{q\sqrt{\pi}} e^{-x^2/q^2} \right) \left(\frac{1}{q\sqrt{\pi}} e^{-y^2/q^2} \right)$$
$$= E_0 e^{ikz} \frac{w_0^2}{q^2} e^{-(x^2+y^2)/q^2}$$

Solved!

Field remains Gaussian

But complicated since q is complex

Calculate $|E(\mathbf{r})|^2$

Use $\frac{1}{q^2} = \frac{1}{w_0^2 + i2z/k}$

$$= \frac{w_0^2 - i2z/k}{w_0^4 + 4z^2/k^2}$$

$$\equiv \frac{1}{w^2} \left(1 - i \frac{2z}{kw_0^2} \right)$$

for

$$w^2 = w_0^2 + \frac{4z^2}{k^2 w_0^2} = \frac{|q|^4}{w_0^2}$$

Then

$$|E(\mathbf{r})|^2 = |E_0|^2 \frac{w_0^4}{|q|^4} e^{-2(x^2+y^2)/w^2}$$
$$= |E_0|^2 \frac{w_0^2}{w^2} e^{-2(x^2+y^2)/w^2}$$

Irradiance remains Gaussian, but size expands

$$w(z) = \sqrt{w_0^2 + \frac{\lambda^2 z^2}{\pi^2 w_0^2}} \to \frac{\lambda z}{\pi w_0}$$

Divergence angle $\theta = \lambda/\pi w_0 \approx \lambda/a$

feature size a

For large w_0, divergence is slow

Light propagates $\approx$ uniformly

Call solution *Gaussian beam*

always Gaussian profile

More on Gaussian beams later in course

Summary:

- Diffraction due to wave nature of light
- Can use Fourier analysis to calculate
  - $E(x, y, 0) \rightarrow \mathcal{A}(k_x, k_y)$
  - know $k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$
- Transfer function: $E(z) \rightarrow E(z + d)$

  $\mathcal{A}_d = \mathcal{H}_d \mathcal{A}$

  $\mathcal{H}_d = e^{ik_z d}$
- Fresnel approximation: expansion of $\mathcal{H}_d$
- Apply to Gaussian beam ≈ laser beam