Huygens-Fresnel Theory and Fraunhofer Diffraction

Last time, started looking at diffraction
 = spreading of light after aperture

Applied Fourier techniques to calculate

Continue discussion today
 Focus on approximations and interpretation

Outline:

- Review
  - Connect to general systems methods
- Propagation as convolution
  - Huygens-Fresnel theory
- Fraunhofer diffraction

Next time: applications of diffraction

Review

Problem:

Given $E = A(x, y)$ in plane $z = 0$

Want to calculate $E(x, y, z)$

Key insight:

If $A(x, y)$ is a harmonic function, know how to solve

Say $A(x, y) = E_0 e^{i(k_x x + k_y y)}$

Then plane wave $E(\mathbf{r}) = E_0 e^{i(k_x x + k_y y + k_z z)}$

satisfies $E(x, y, 0) = A(x, y)$ for any k_z

Want $E(\mathbf{r})$ = solution of wave equation

$\Rightarrow$ require $k^2 = k_x^2 + k_y^2 + k_z^2 = n^2\omega^2/c^2$

(assume n and ω known)

So need $k_z = \pm\kappa = \pm\sqrt{k^2 - k_x^2 - k_y^2}$

$+$ for wave travelling toward positive z

So we can solve problem for harmonic $A(x,y)$

Wave equation is linear:

If $A(x,y)$ = sum of harmonic funcs

Then solution = sum of plane waves

Fourier transform:

Express any A as sum of harmonic funcs:

$$A(x,y) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y)} \, dk_x \, dk_y$$

Each term $e^{i(k_x x + k_y y)} \rightarrow e^{i(k_x x + k_y y + \kappa z)}$

So general solution is

$$E(x,y,z) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) e^{i(k_x x + k_y y + \kappa z)} \, dk_x \, dk_y$$

with $\kappa = \sqrt{k^2 - k_x^2 - k_y^2}$

Or write as

$$E(x,y,z) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y) \mathcal{H}(k_x, k_y) e^{i(k_x x + k_y y)} \, dk_x \, dk_y$$

with $\mathcal{H} = e^{iz\sqrt{k^2 - k_x^2 - k_y^2}} \equiv$ transfer function

More on transfer functions:

Think of propagation as "system"

$A(x,y)$ = input to system

$E(x,y,z)$ = output

Or more generally:

$E(x,y,z_0) \equiv A(x,y)$ = input

$E(x,y,z_0+d) \equiv A_d(x,y)$ = output

Transfer function relates output to input

General definition:

$$\mathcal{H}_d(k_x,k_y) = \frac{\text{output}}{\text{input}} = \frac{A_d(x,y)}{A(x,y)}$$

for *particular case* when input is harmonic

$$A(x,y) = E_0 e^{i(k_x x + k_y y)}$$

Here $A_d = E_0 e^{i(k_x x + k_y y + \kappa d)}$ so $\mathcal{H}_d = e^{i\kappa d}$

Question: In the definition above, it looks like $\mathcal{H}$ should depend on x and y. Why doesn't it?

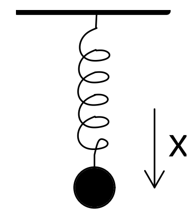
Transfer function idea applies to any linear system

Harmonic input always easy to solve

Example: mass on spring

input = force on mass $F(t)$

output = steady state position $x(t)$



Or (voltage, current) in RLC circuit

Or (drive, response) of NMR sample

Simple harmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f(t)$$

where ω_0 = natural oscillation frequency

γ = damping coefficient

$f(t) = F(t)/m$

Solve for arbitrary $f(t)$

Actually kind of hard

But solving for $f = f_0 e^{-i\omega t}$ is easy

Try $x = x_0 e^{-i\omega t}$:

Get $-\omega^2 x_0 - i\gamma\omega x_0 + \omega_0^2 x_0 = f_0$

So $x(t) = \frac{1}{\omega_0^2 - i\gamma\omega - \omega^2} f(t)$

Then transfer function is

$$\mathcal{H}(\omega) = \frac{x(t)}{f(t)} = \frac{1}{\omega_0^2 - i\gamma\omega - \omega^2}$$

For general $f(t)$, have

$$f(t) = \frac{1}{2\pi} \int \mathcal{F}(\omega) e^{-i\omega t} d\omega$$

So general solution is

$$x(t) = \frac{1}{2\pi} \int \mathcal{F}(\omega) \mathcal{H}(\omega) e^{-i\omega t} d\omega$$

Integral might be hard to do analytically,
but problem is solved

Same basic method as in optics

Back to optics...

Use approximations to simplify solution

Start with Fresnel approximation:

$$\kappa = \sqrt{k^2 - k_x^2 - k_y^2} \approx k - \frac{k_x^2 + k_y^2}{2k}$$

Valid for $k_x, k_y \ll k$: small propagation angles θ

Then $\mathcal{H}_d(k_x, k_y) \approx e^{ikd}\, e^{-id(k_x^2+k_y^2)/2k}$

Used to solve Gaussian beam problem

Convolutions

Another way to look at solution

Have

$$A_d(x, y) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y)\mathcal{H}_d(k_x, k_y)e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

Fourier transform says

$$A_d(x, y) = \frac{1}{(2\pi)^2} \iint \mathcal{A}_d(k_x, k_y)e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

So $\mathcal{A}_d = \mathcal{A}\mathcal{H}_d$

Recall convolution theorem:

If $F(\omega) = F_1(\omega)F_2(\omega)$ then

$$f(t) = \int_{-\infty}^{\infty} f_1(T) f_2(t-T)\, dT$$

We have 2D version:

$$\mathcal{A}_d(k_x, k_y) = \mathcal{A}(k_x, k_y)\mathcal{H}(k_x, k_y)$$

so $A_d(x, y) = \iint A(X, Y) h_d(x - X, y - Y)\, dX\, dY$

where $h_d(x, y)$ = inverse transform of $\mathcal{H}_d$

Let's work out h_d in Fresnel approximation

Have $h_d(x, y) =$

$$\frac{1}{(2\pi)^2} \iint e^{ikd} e^{-id(k_x^2+k_y^2)/2k} e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

$$= e^{ikd} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-idk_x^2/2k}\, e^{ik_x x}\, dk_x \right) \times \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-idk_y^2/2k}\, e^{ik_y y}\, dk_y \right)$$

Gaussian transforms

$$e^{-k_x^2 q^2/4} \rightarrow \frac{1}{q\sqrt{\pi}} e^{-x^2/q^2}$$

Here $q^2 = 2id/k$

So $h(x, y) = \dfrac{e^{ikd}}{\pi q^2} e^{-(x^2+y^2)/q^2}$

$$= \frac{ke^{ikd}}{2i\pi d} e^{-k(x^2+y^2)/(2id)}$$

$$\boxed{h(x, y) = -i\frac{e^{ikd}}{\lambda d} e^{ik(x^2+y^2)/2d}}$$

where $\lambda = 2\pi/k$

Question: Do the units (m^{-2}) for h make sense?

Interesting fact:

Write h_d as $h_d(x, y) = -\dfrac{i}{\lambda d} e^{ik\left(d + \frac{x^2+y^2}{2d}\right)}$

Note that $d + \dfrac{x^2 + y^2}{2d} \approx \sqrt{x^2 + y^2 + d^2}$

for $d \gg x, y$

But $d = z - z_0$, so $\boxed{h_d(x, y) = -\dfrac{i}{\lambda}\dfrac{e^{ikr}}{r}}$

for $r = \sqrt{x^2 + y^2 + (z - z_0)^2}$
and $d \approx r$ in denominator

So $h_d(x,y) \approx$ spherical wave centered at $(0,0,z_0)$

Look again at expression for fields:

$$A_d(x,y) = \iint A(X,Y) h_d(x-X, y-Y)\, dX\, dY$$

Set $z_0 = 0$ for simplicity

Then $A_d(x,y) = E(x,y,z)$ and $A(x,y) = E(x,y,0)$

So

$$E(x,y,z) = \iint E(X,Y,0) h_z(x-X, y-Y)\, dX\, dY$$

$$E(x,y,z) = \iint E(X,Y,0) h_z(x-X, y-Y)\, dX\, dY$$

or $$\boxed{E(\mathbf{r}) = -\frac{i}{\lambda} \iint E(\mathbf{R}) \frac{e^{ik|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}\, dX\, dY}$$

for $\mathbf{r} = (x,y,z)$, $\mathbf{R} = (X,Y,0)$

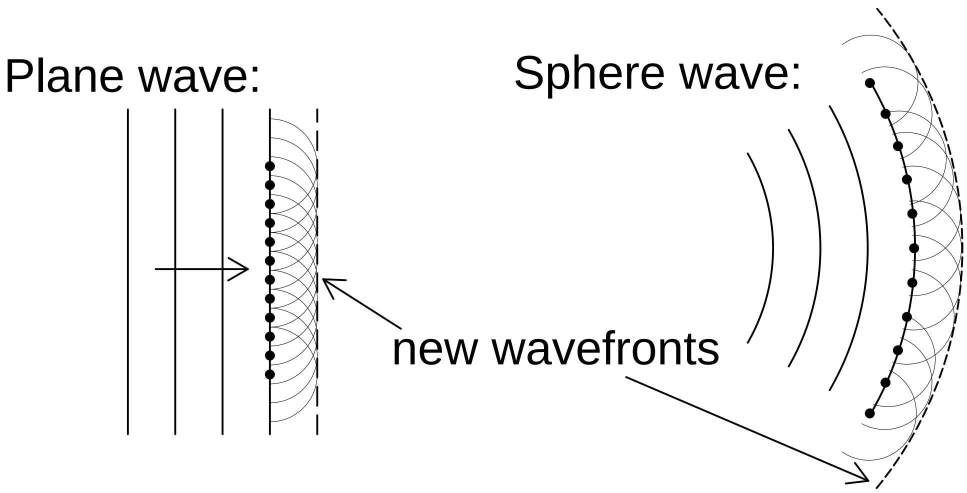
Interpretation:

- Each point $\mathbf{R}$ in input plane acts as source of spherical wave amplitude $\propto E(\mathbf{R})$

- Total field at $\mathbf{r}$ in output plane = sum of spherical waves from different $\mathbf{R}$'s

Fresnel-Huygens picture (Hecht 4.4.2)

Each point on wavefront = source of new waves
- call new waves "Huygens's wavelets"



Completely different way of looking at diffraction:
no Fourier transform involved

Huygens's principle is often starting point
Hecht Ch 10, and other texts
Offers easier, intuitive picture

Disadvantages:

- Not obvious why it's true
- Doesn't give amplitude factor $-i/\lambda$
- Often Fourier space integral is easier
- Easily confused with scattering ideas

Scattering and Huygens

In a real medium with $n \neq 1$,
each point *is* source of new wave E_{scat}
(and $E_{\text{scat}} \approx$ spherical wave)

But total field = $E_{\text{inc}} + E_{\text{scat}}$

In Huygens picture, E_{scat} is total field

Imagine vacuum is dense elastic medium:
Disturbance comes from motion of nearby points
Sum over nearby points = sum over E_{scat}
= total field

But vacuum is not elastic medium!

Light = EM wave, only sources are charges
no charge in vacuum, so no source

Confusing: Huygens picture not "real"
- just a way of interpreting an integral

Interesting that Maxwell equations suggest
that vacuum acts "sort of like" a medium

But not fundamental

We'll stick to Fourier picture

Still

$$A_d(x, y) = \iint A(X, Y)h(x - X, y - Y)\, dX\, dY$$

is very useful

In systems language,

$h =$ *impulse-response function*

If $A(X, Y) = E_0\delta(X, Y)$, then

$$A_d(x, y) = E_0 h(x, y)$$

$h(x, y) =$ field produced by point source

makes sense, = spherical wave

General system:

impulse-response function and transfer function

are Fourier transforms

For harmonic oscillator example:

$$h(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{\omega'} e^{-\gamma t/2} \sin(\omega' t) & (t > 0) \end{cases}$$

for $\omega' = \sqrt{\omega_0^2 + \gamma^2/4}$ = shifted oscillation frequency

$h(t) =$ free decay of oscillator after kick at $t = 0$

Secret for physics grad students:

- impulse-response function
  = Green's function
- transfer function = propagator

Important for EM, condensed matter, quantum field theory

Fraunhofer Diffraction (Hecht 10.2)

Drawback of results so far:
integrals are too hard

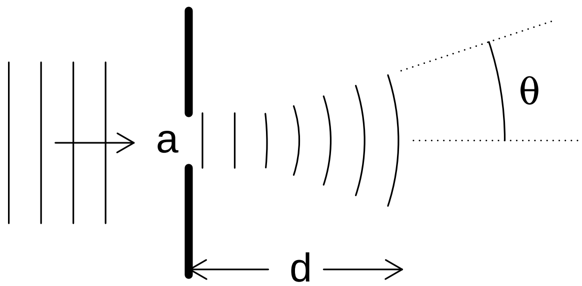
Even in Fresnel approximation, only Gaussians easy

Develop simpler approximation:
Fraunhofer diffraction

Idea:
Suppose aperture transmits region size a
(ie, hole in opaque screen)

Diffraction pattern spreads at angle $\theta \approx \lambda/a$

Spreads distance $\approx \theta d$

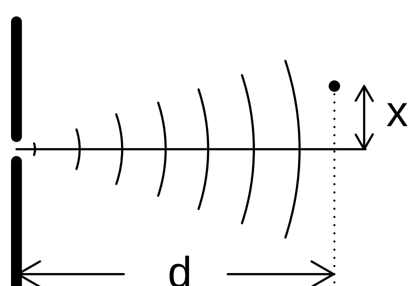


For $\theta d \ll a$, diffraction not significant
use ray optics

For $\theta d \approx a$: complicated
Need to use Fresnel integrals

For $\theta d \gg a$: simplifies again

Pattern spread over large area

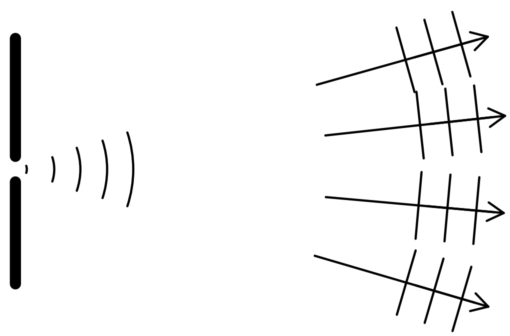


Light reaching position x
corresponds to angle $\theta_x = x/d$

But angle θ_x corresponds to wavenumber $k_x = \theta_x k$

$\Rightarrow$ At large d, wave vector $\mathbf{k}$ maps to position

Or: field = sum of plane waves
for $d \to \infty$, plane waves separate



Expect field at (x, y) corresponds to
Fourier component $(k_x, k_y) = (kx/d, ky/d)$

Make precise:

$$A_d(x,y) = \iint A(X,Y) h(x - X, y - Y)\, dX\, dY$$

$$= -\frac{i}{\lambda d} e^{ikd} \iint A(X,Y) e^{ik[(x-X)^2 + (y-Y)^2]/2d}\, dX\, dY$$

Consider exponent:

$$i\frac{k}{2d}\left(x^2 + y^2 + X^2 + Y^2 - 2xX - 2yY\right)$$

Know $X^2 + Y^2 < a^2$ for aperture size a

If $ka^2/2d \ll 1$, can neglect
or $d \gg ka^2/2 \approx a^2/\lambda$

Then have

$$A_d(x,y) = -\frac{i}{\lambda d} e^{ikd}\, e^{ik(x^2+y^2)/2d} \times \iint A(X,Y) e^{-ik(xX+yY)/d}\, dX\, dY$$

But notice that

$$\mathcal{A}(k_x,k_y) = \iint A(X,Y) e^{-i(k_x X + k_y Y)}\, dX\, dY$$

integral has same form, $k_x \to kx/d$, $k_y \to ky/d$

So

$$\boxed{A_d(x,y) = -\frac{i}{\lambda d} e^{ikd}\, e^{ik(x^2+y^2)/2d} \mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)}$$

Phase shifts drop out in irradiance:

$$|A_d(x,y)|^2 = \frac{1}{\lambda^2 d^2} \left| \mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right) \right|^2$$

But phase shift makes sense too:

$$\frac{1}{d} e^{ik\left[d + (x^2+y^2)/2d\right]} \approx \frac{e^{ikr}}{r}$$

as before

Diffracted field looks like spherical wave
modulated by $\mathcal{A}$

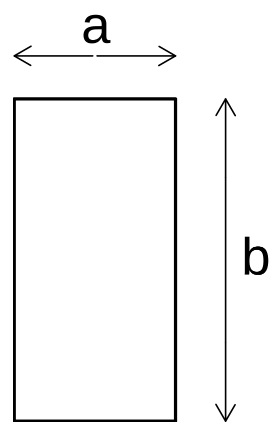
Question: In spherical wave expression above, what point in aperture is r supposed to be measured from?

Finally lets us calculate something

Consider plane wave incident on rectangular slit

Then

$$A(x, y) = \begin{cases} E_0 & (|x| < a/2, |y| < b/2) \\ 0 & (\text{otherwise}) \end{cases}$$



Calculate diffraction pattern for large d

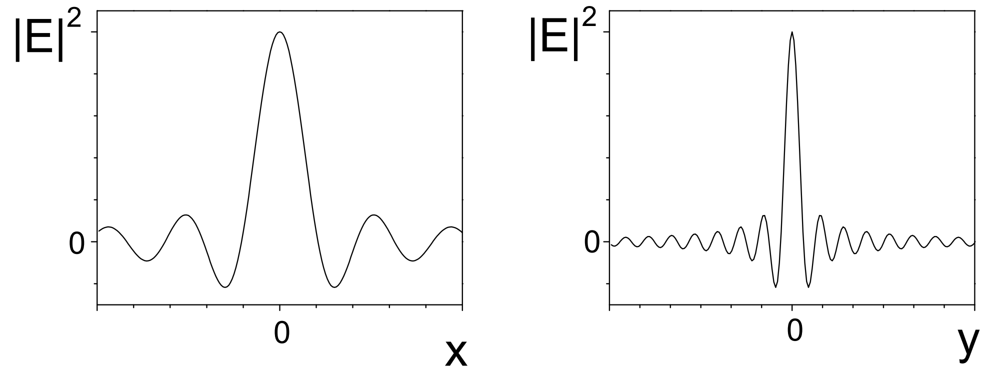
Know

$$\mathcal{A}(k_x, k_y) = E_0 ab \, \text{sinc}\left(\frac{k_x a}{2}\right) \text{sinc}\left(\frac{k_y b}{2}\right)$$

So

$$|A_d(x, y)|^2 = \frac{a^2 b^2}{\lambda^2 d^2} |E_0|^2 \, \text{sinc}^2\left(\frac{kxa}{2d}\right) \text{sinc}^2\left(\frac{kyb}{2d}\right)$$

Question: Remind us of the definition of sinc one more time?



Have sinc$(kxa/2d) = 0$ when $kxa/2d = n\pi$

$n =$ nonzero integer

So width of central peak $\Delta x = \dfrac{4\pi d}{ka} = \dfrac{2\lambda d}{a}$

and $\Delta y = \dfrac{2\lambda d}{b}$

More on Fraunhofer diffraction next time

Summary:

- Convolution theorem: impulse-response function
  - Fourier transform of transfer function
- Interpret as Huygens-Fresnel theory
  - Each point in aperture generates spherical wave
- Large d limit: Fraunhofer diffraction
  - Diffraction pattern directly from transform