Applications of Diffraction

Last time, developed Fraunhofer diffraction

At large distances, diffracted field
$\propto$ transform of aperture function

Each Fourier component propagates in different direction

Today, explore Fraunhofer further

Outline:

- Summary of diffraction regimes
- Diffraction from an array
- Circular apertures
- Diffraction and lenses

Next time, consider fancier applications
- Fourier optics
- Holography

Diffraction Regimes

Several ways to study diffraction

Choice of method depends on
- wavelength λ
- aperture size a
- propagation distance d
- propagation angle θ

If $d \ll a^2/\lambda$, use ray optics
Aperture produces geometric shadow

Get diffraction effects near sharp edges
Noticeable at $d \approx$ few cm

For large d and $\theta \gtrsim (\lambda/d)^{1/4}$,
need "exact" expression

$$E(\mathbf{r}) = \frac{1}{(2\pi)^2} \iint \mathcal{A}(k_x, k_y)\mathcal{H}(k_x, k_y)e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

with $\mathcal{H}(k_x, k_y) = e^{id\sqrt{k^2 - k_x^2 - k_y^2}}$

Still approximate, fails for $\theta \gtrsim 1$:
- Ignores vector nature of $\mathbf{E}$
- Don't really know $A(x, y)$:
  Depends on aperture thickness, material

Large angle effects very hard
Numerically solve Maxwell equations

For $\theta \lesssim (\lambda/d)^{1/4}$:

If $d \sim a^2/\lambda$, use Fresnel approximation

Either:

$$\mathcal{H}(k_x, k_y) = e^{ikd}\, e^{-id(k_x^2+k_y^2)/2k}$$

with Fourier form

or convolution form:

$$E(\mathbf{r}) = \iint A(X,Y)h(x-X, y-Y)\, dX\, dY$$

with $h(x,y) = -i\dfrac{e^{ikd}}{\lambda d}\, e^{ik(x^2+y^2)/2d}$

If $d \gg a^2/\lambda$, use Fraunhofer approximation:

$$E(\mathbf{r}) = -\frac{i}{\lambda d} e^{ikd}\, e^{ik(x^2+y^2)/2d} \mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)$$

Simplest form of diffraction

Also called “far-field” diffraction

Extra important in lens systems

- later today

Example: square aperture

size $a = 1$ mm

$\lambda = 500$ nm

Then $a^2/\lambda = 2$ m:

- $d < 0.2$ m, use ray optics
- 0.2 m $< d < 20$ m, use Fresnel
- $d > 20$ m, use Fraunhofer

At $d = 2$ m, maximum angle for Fresnel

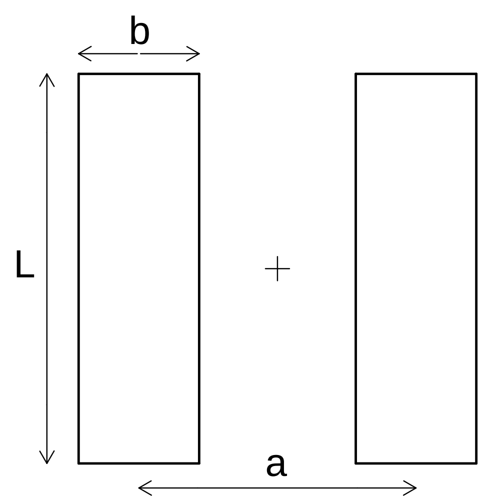
$\approx (\lambda/d)^{1/4} \approx 20$ mrad $\approx 1^\circ$

Corresponds to distance $x = 4$ cm

observed pattern size $\sim$ few mm

Fraunhofer Examples

First: two slits (Hecht 10.2.2)



Slit width $(x) = b$

Height $(y) = L$

Center separation $= a$

(Hecht's notation)

Say $x = 0$, $y = 0$ in center of pair

Need $\mathcal{A}(k_x, k_y)$

Don't bother doing integral:

Know $\mathcal{A}$ for single slit centered at $x = y = 0$ is

$$\mathcal{A}_1 = bL\,\mathrm{sinc}(k_x b/2)\,\mathrm{sinc}(k_y L/2)$$

From translation property,

$$f(x - X) \to e^{-ik_x X}\mathcal{F}(k_x)$$

so slit at $x = a/2$ has transform

$$\mathcal{A}_1' = bLe^{-ik_x a/2}\,\mathrm{sinc}(k_x b/2)\,\mathrm{sinc}(k_y L/2)$$

Slit at $x = -a/2$ has transform

$$\mathcal{A}_2' = bLe^{ik_x a/2}\,\mathrm{sinc}\left(\frac{k_x b}{2}\right)\mathrm{sinc}\left(\frac{k_y L}{2}\right)$$

Since transform is linear, pair gives

$$\mathcal{A}(k_x, k_y) = bL\left(e^{ik_x a/2} + e^{-ik_x a/2}\right) \times \mathrm{sinc}\left(\frac{k_x b}{2}\right)\mathrm{sinc}\left(\frac{k_y L}{2}\right)$$

$$= 2bL\cos\left(\frac{k_x a}{2}\right)\mathrm{sinc}\left(\frac{k_x b}{2}\right)\mathrm{sinc}\left(\frac{k_y L}{2}\right)$$

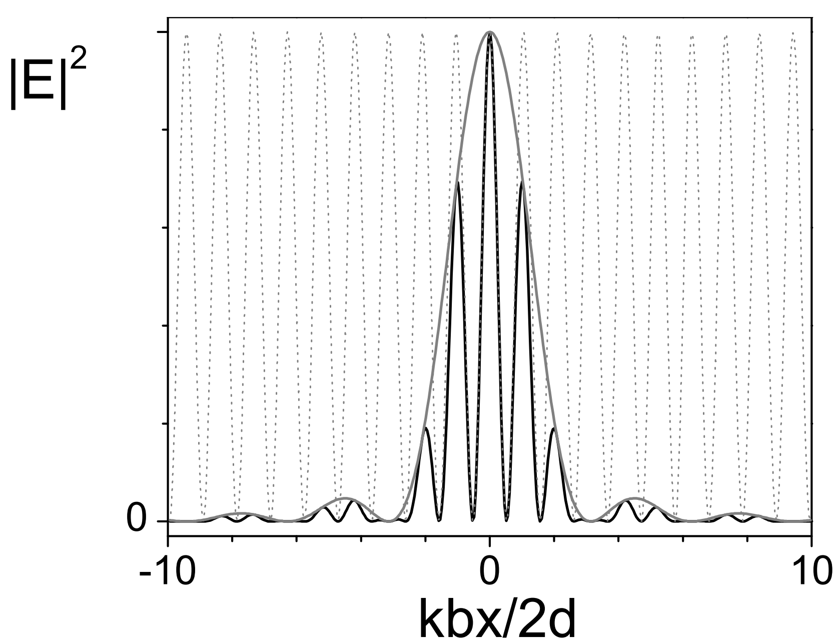
Then diffracted field is

$$E(x,y) = \frac{1}{\lambda d} C \mathcal{A}\left(\frac{kx}{d}, \frac{ky}{d}\right)$$
$$= C\frac{2bL}{\lambda d} \cos\left(\frac{kxa}{2d}\right) \text{sinc}\left(\frac{kxb}{2d}\right) \text{sinc}\left(\frac{kyL}{2d}\right)$$

for $C = -ie^{ikd}\, e^{ik(x^2+y^2)/2d}$, with $|C| = 1$

Question: Why does the cosine factor depend on x but not y?

Plot vs. x for $a = 3b$



Like pattern for single slit $\times 2\cos(kxa/2d)$

Recall lecture 13:
Interference pattern from two point sources

$$E(x) = 2E_1 \cos(kxa/2d)$$

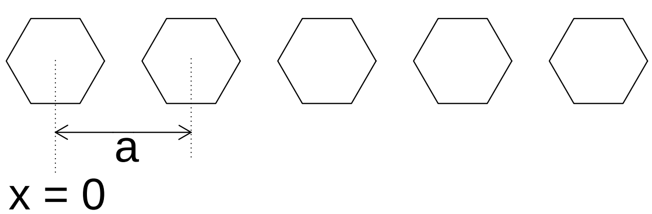
E_1 = field from single source

$\Rightarrow$ Get product of two-point pattern
and single slit pattern

Diffraction from an Array (Hecht 10.2.3)

What if there were N slits?

Or generalize:
suppose array of N identical apertures



Each aperture described by $A_1(x, y)$
Centers at $x = na$ for $n = 0$ to $N - 1$

Individual apertures have transform $\mathcal{A}_1(k_x, k_y)$
(hard to do for hexagon)

Given $\mathcal{A}_1$, what is total field?

Have $A(x,y) = \sum_{n=0}^{N-1} A_1(x - na, y)$

Using linearity and translation:

$$\mathcal{A}(k_x, k_y) = \sum_{n=0}^{N-1} e^{-ink_x a} \mathcal{A}_1(k_x, k_y)$$
$$= \mathcal{A}_1(k_x, k_y) \sum_{n=0}^{N-1} e^{-ink_x a}$$

Define

$$\mathcal{P}(k_x) = \sum_{n=0}^{N-1} e^{-ink_x a}$$

independent of individual aperture shape

So Fraunhofer diffraction field is

$$A_d(x,y) \propto \mathcal{A}_1\left(\frac{kx}{d}, \frac{ky}{d}\right) \mathcal{P}\left(\frac{kx}{d}\right)$$

envelope $\mathcal{A}_1$ modulated by $\mathcal{P}$

Calculate $\mathcal{P}$: for $\alpha = e^{-ik_x a}$,

$$\mathcal{P} = \sum_{n=0}^{N-1} \alpha^n$$

Geometric series:

$$\mathcal{P} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1}$$

$$\alpha\mathcal{P} = \alpha + \alpha^2 + \cdots + \alpha^N$$

So $\mathcal{P} - \alpha\mathcal{P} = 1 - \alpha^N = (1-\alpha)\mathcal{P}$

$$\mathcal{P} = \frac{1-\alpha^N}{1-\alpha} = \frac{1 - e^{-iNk_x a}}{1 - e^{-ik_x a}}$$

Rewrite

$$\mathcal{P}(k_x) = \left(\frac{e^{-iNk_x a/2}}{e^{-ik_x a/2}}\right)\left(\frac{e^{iNk_x a/2} - e^{-iNk_x a/2}}{e^{ik_x a/2} - e^{-ik_x a/2}}\right)$$

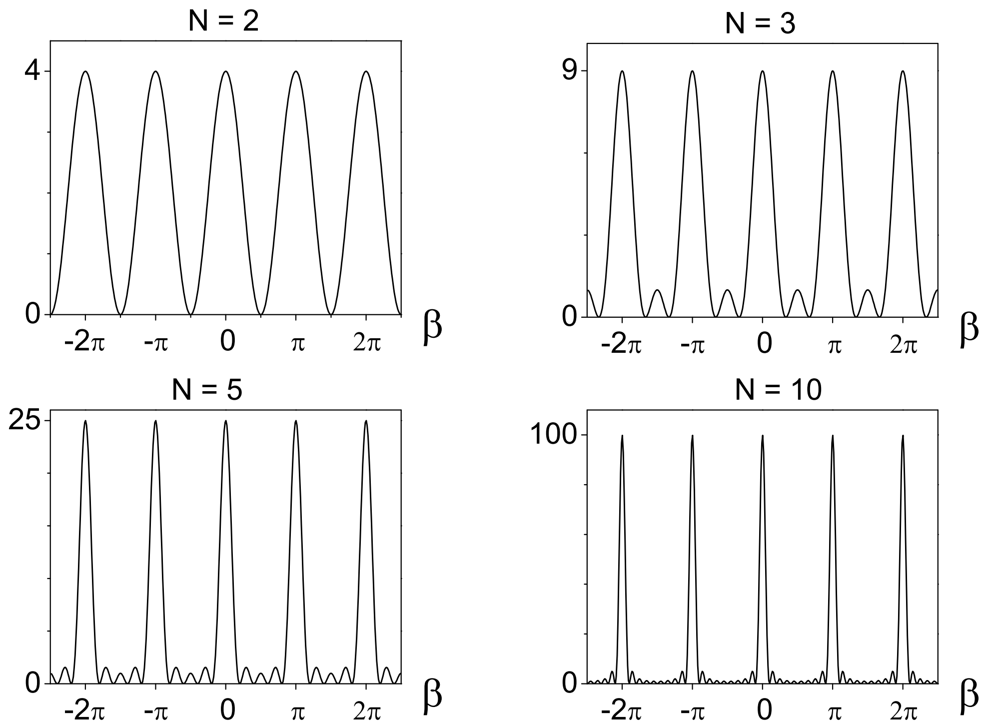
$$= e^{-ik_x[(N-1)a/2]}\left[\frac{\sin(Nk_x a/2)}{\sin(k_x a/2)}\right]$$

$$= e^{-ik_x x_m}\,\frac{\sin(Nk_x a/2)}{\sin(k_x a/2)}$$

where $x_m = \dfrac{(N-1)}{2}\,a =$ center of pattern

Question: What is $\mathcal{P}(0)$?

Plot $\sin^2(N\beta)/\sin^2(\beta)$:



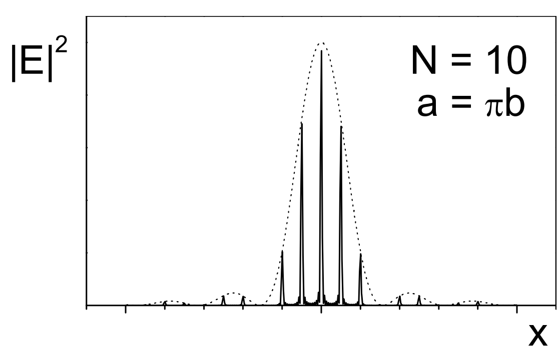
Peaks located at $\beta = n\pi$

Get sharper as N increases

width $\Delta\beta = 2\pi/N$

Remember, $\mathcal{P}$ multiplies single aperture pattern

Ten rectangular slits:



Sharp lines useful for spectroscopy

Peaks at $\beta = \frac{kxa}{2d} = m\pi$

or $x = \frac{2\pi md}{ka} = \frac{m\lambda d}{a}$: depends on λ

If d, a known, use to determine λ

Works about the same even for $a, b \approx \lambda$
(Fresnel approx not valid)

Get peaks at angles $\boxed{\sin\theta = \frac{m\lambda}{a}}$

m = *order* of maximum

Demo: diffraction grating

Grating $a = 1.7\ \mu$m
specify $1/a = 600$ lines/mm

Light: Hg lamp
$\lambda = 578$ nm (yellow)
$\lambda = 546$ nm (green)
$\lambda = 435$ nm (blue)

Grating is highly dispersive
-better than prisms

Circular Apertures (Hecht 10.2.5)

Say aperture is circular hole, radius a

$$A(x,y) = \begin{cases} E_0 & \left(\sqrt{x^2+y^2} < a\right) \\ 0 & \text{(else)} \end{cases}$$

Need to know Fourier transform

$$\mathcal{A}(k_x, k_y) = \iint A(x,y) e^{-i(k_x x + k_y y)}\, dx\, dy$$

Can't separate into two 1D transforms

- need to work out integral

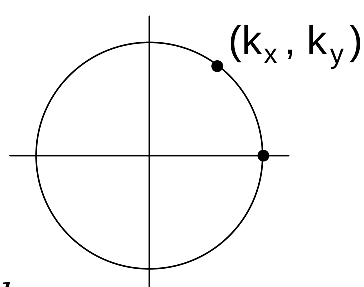
Use polar coordinates

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$

Expect $\mathcal{A}$ symmetric in (k_x, k_y)

So solve for $k_y = 0$, then use

$$\mathcal{A}(k_x, k_y) = \mathcal{A}\left(\sqrt{k_x^2 + k_y^2}, 0\right)$$



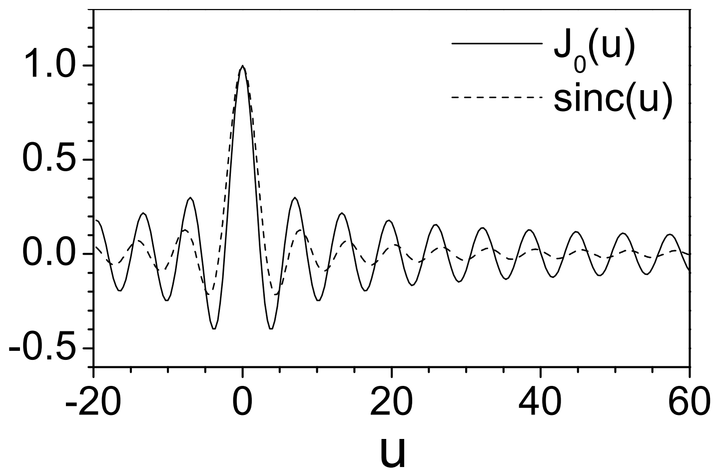
$$\mathcal{A}(k_x, 0) = \int_0^a \int_0^{2\pi} e^{-ik_x \rho \cos\phi} \rho\, d\phi\, d\rho$$

Integral not elementary

Have $\int_0^{2\pi} e^{-ik_x\rho\cos\phi} d\phi = 2\pi J_0(k_x\rho)$

J_0 = Bessel function

Like sinc, but not exactly

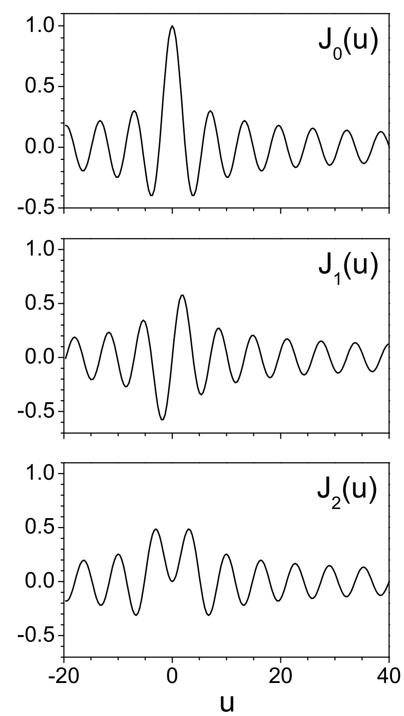


Bessel Function Primer

Family of functions:

$J_m(u) =$

$$\frac{1}{2\pi i^m}\int_0^{2\pi} e^{i(m\phi+u\cos\phi)}\, d\phi$$



Fairly common functions
(after trig, exp)

Summarize properties

Solutions of Bessel's equation:

$$u^2 J_m'' + u J_m' + (u^2 - m^2) J_m = 0$$

Power series:

$$J_m(u) = \left(\frac{u}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{u}{2}\right)^{2n}$$

Large u expansion:

$$J_m(u) \to \left(\frac{2}{\pi u}\right)^{1/2} \cos\left[u - \frac{(2m+1)\pi}{4}\right]$$

Orthogonality:

$$\int_0^\infty u J_m(\alpha u) J_m(\beta u)\, du = \frac{1}{\alpha}\delta(\alpha - \beta)$$

Derivative relation:

$$\frac{d}{du}\left[u^m J_m(u)\right] = u^m J_{m-1}(u)$$

Think of $\approx$ cosine (even m)
or sine (odd m)

Apply to diffraction problem

$$\mathcal{A}(k_x, 0) = 2\pi \int_0^a \rho J_0(k_x \rho)\, d\rho$$

Set $u = k_x \rho$

$$\mathcal{A}(k_x, 0) = \frac{2\pi}{k_x^2} \int_0^{k_x a} u J_0(u)\, du$$

From derivative relation

$$u J_0(u) = \frac{d}{du}[u J_1(u)]$$

so

$$\int u J_0(u)\, du = u J_1(u)$$

So transform is

$$\mathcal{A}(k_x, 0) = \frac{2\pi}{k_x^2} \left[u J_1(u)\right]\Big|_0^{k_x a}$$

or

$$\boxed{\mathcal{A}(k_x, k_y) = \frac{2\pi a}{k_\rho} J_1(k_\rho a)}$$

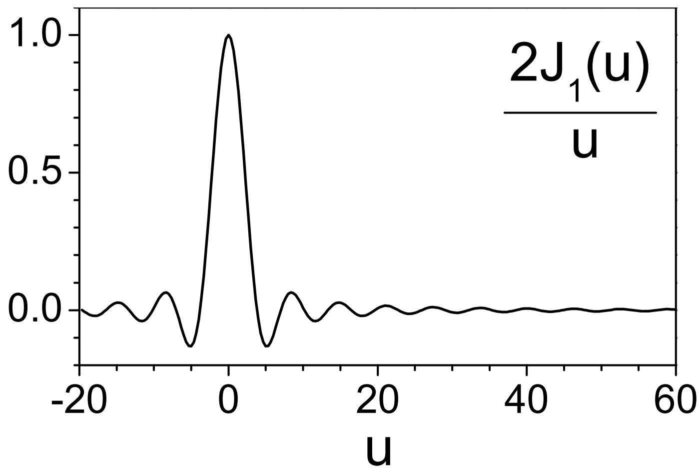
for $k_\rho \equiv \sqrt{k_x^2 + k_y^2}$

Note, for small k_ρ, $J_1(k_\rho a) \to k_\rho a/2$

Write

$$\mathcal{A}(k_x, k_y) = \pi a^2 \left[\frac{2 J_1(k_\rho a)}{k_\rho a}\right]$$

Where $\left[\dfrac{2J_1(k_\rho a)}{k_\rho a}\right]$ is like sinc function



So Fraunhofer pattern is

$$|E(x,y)|^2 = |E_0|^2 \left(\frac{2\pi a}{\lambda k \rho}\right)^2 J_1\left(\frac{k\rho a}{d}\right)^2$$

$$= |E_0|^2 \frac{a^2}{\rho^2} J_1\left(\frac{k\rho a}{d}\right)^2$$

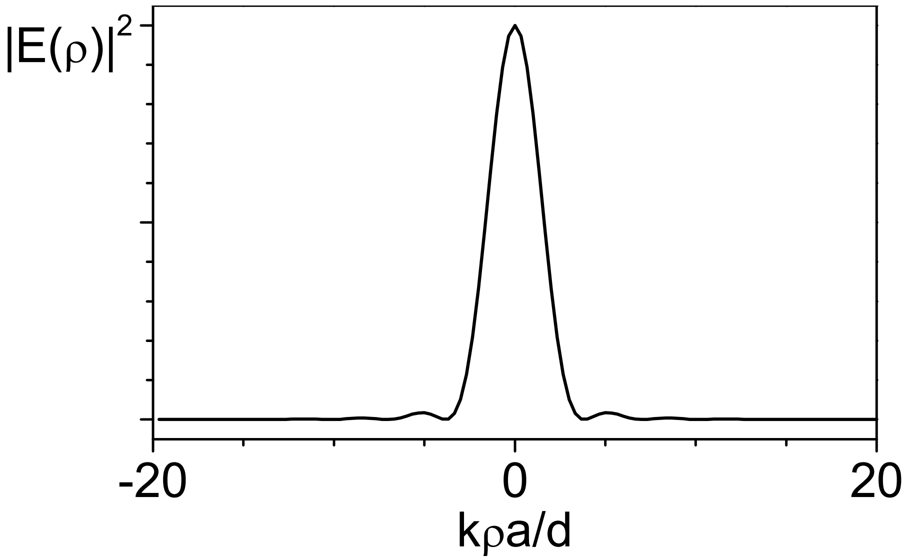
First zero at $k\rho a/d = 3.83$

$$\rho = 3.83\frac{d}{ak} = 1.22\frac{\lambda d}{a}$$

Diffraction angle $\theta = \rho/d = 1.22\lambda/a$

$\theta \approx \lambda/a$ as always

Called Airy pattern:



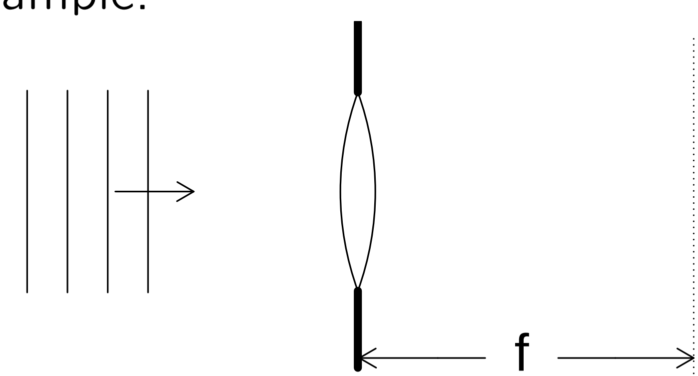
Question: The secondary maxima for a circular aperture are smaller than for a square aperture. Why?

Diffraction and Lenses

Frauhofer only valid for very large d

Usually observed using lenses:
image $d = \infty$ onto focal plane

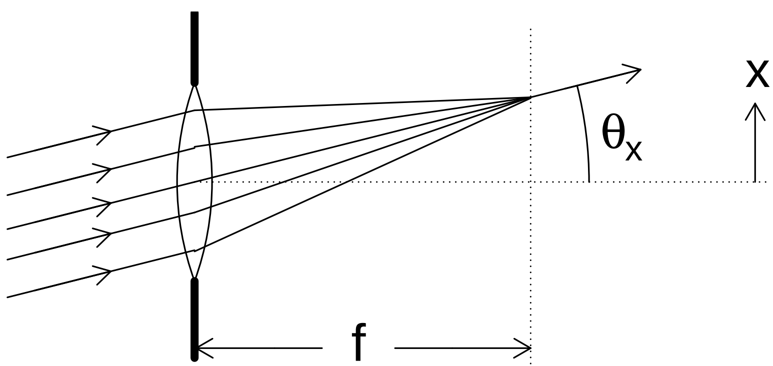
Example:



Plane wave incident on lens diameter D

Without lens, get Airy pattern at ∞

Point $x = \theta_x d$ at ∞
maps to $x = \theta_x f$ in focal plane



So expect $E(x, y) \propto \mathcal{A}(kx/f, ky/f)$

At focal plane of lens

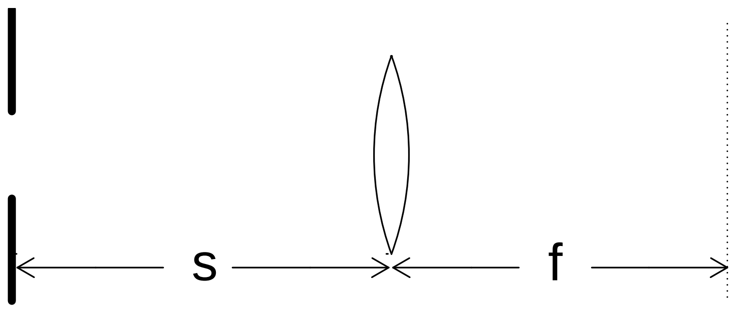
$$|E(x, y)|^2 = |E_0|^2 \frac{D^2}{4\rho^2} J_1\left(\frac{k\rho D}{2f}\right)^2$$

(using $D = 2a$)

Gives spot diameter $\boxed{1.22\,\frac{\lambda f}{D}}$
= resolution limit of lens

Same result cited in discussion of aberrations

More generally,
lens gives you Fraunhofer pattern for any object:



At object plane, $E = A(x,y)$

Set by aperture or other condition

Assume diffraction angle $\ll$ (lens diameter)/s

- so lens aperture unimportant

At focal plane

$$\boxed{E(x,y) = \frac{C}{\lambda f}\,\mathcal{A}\left(\frac{kx}{f},\frac{ky}{f}\right)}$$

with phase factor

$$C = -ie^{ik(s+f)}\,\text{exp}\left[-i\frac{(x^2+y^2)(s-f)}{2k}\right]$$

Result valid within Fresnel approximation

(Derivation Saleh and Teich 4.2B)

Note only phase depends on s
Object has same far-field pattern for any position

Focal plane of lens is special
Sometimes called "transform plane"

Convenient way to see Fraunhofer pattern
other applications next time

Summary:

- Diffraction approx controlled by $a^2/\lambda d$
  But no approximations for large θ
- Array gives single-object pattern,
  modulated by grating function
- Grating: sharp peaks for large N
- Circular aperture $\rightarrow$ Bessel function
  pattern radius = $1.22\,\lambda a/d$
- Lens: Fraunhofer pattern $\rightarrow$ focal plane
  Mostly $d \rightarrow f$ everywhere