Electromagnetic Theory (Hecht Ch. 3)

Last time, talked about waves in general

- wave equation: $\nabla^2\psi(\mathbf{r},t) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

ψ = amplitude of disturbance of medium

For light, "medium" = EM field

This time:

- derive wave equation from Maxwell's Equations
- study properties of EM waves in vacuum

Next time:

- consider EM waves in matter

Maxwell's Equations:

Basic postulates of electromagnetism

Physical quantities:

Electric field	$\mathbf{E}(\mathbf{r},t)$	(Volts/m)
Magnetic field	$\mathbf{B}(\mathbf{r},t)$	(Tesla)
Charge density	$\rho(\mathbf{r},t)$	(Coulombs/m^3)
Current density	$\mathbf{J}(\mathbf{r},t)$	(Amperes/m^2)

Gauss's Laws (charge produces a field):

$$\oiint \mathbf{E}\cdot\mathbf{dS} = \frac{1}{\epsilon_0}\iiint \rho\, dV \qquad \oiint \mathbf{B}\cdot\mathbf{dS} = 0$$

ϵ_0 = permittivity of free space = 8.8 pF/m
(from capacitor measurements)

Faraday's Law (changing $\mathbf{B}$ produces $\mathbf{E}$):

$$\oint \mathbf{E}\cdot\mathbf{d}\ell = -\iint \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{dS}$$

Ampere's Law (current, changing $\mathbf{E}$ produce $\mathbf{B}$):

$$\oint \mathbf{B}\cdot\mathbf{d}\ell = \mu_0 \iint \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)\cdot\mathbf{dS}$$

μ_0 = permeability of free space
= 1.3 μH/m $\equiv 4\pi \times 10^{-7}$ H/m

These are integral form of Maxwell's equations

Will be more useful in *differential form*

Use Gauss's Theorem: for any $\mathbf{F}(\mathbf{r})$

$$\oiint \mathbf{F} \cdot \mathbf{dS} = \iiint \nabla \cdot \mathbf{F} \, dV$$

Here $\nabla \equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$

Also Stoke's Theorem: $\oint \mathbf{F} \cdot \mathbf{d}\ell = \iint (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$

- Analogous to ordinary calculus:

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} \frac{df}{dx} \, dx$$

function on boundary   integral of derivative

Apply to Maxwell:

$$\oiint \mathbf{E} \cdot \mathbf{dS} = \iiint \nabla \cdot \mathbf{E} \, dV$$
$$= \frac{1}{\epsilon_0} \iiint \rho \, dV$$

True for any volume V, so at each point $\mathbf{r}$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

Similarly, get $\nabla \cdot \mathbf{B} = 0$

Also, for any surface S,

$$\oint \mathbf{E} \cdot \mathbf{d}\ell = \iint \nabla \times \mathbf{E} \cdot \mathbf{dS} = -\iint \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$$

so

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and similarly

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

These are differential form of Maxwell's equations

Light Waves (Hecht 3.2)

Light propagates in vacuum: $\rho = \mathbf{J} = 0$

Maxwell equations become:

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

Where's the wave equation?

Derivation of wave equation

Take $\partial/\partial t$ of (4):

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

or

$$\nabla \times \frac{\partial\mathbf{B}}{\partial t} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Then using (3):

$$-\nabla \times (\nabla \times \mathbf{E}) = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

Need to simplify $\nabla \times (\nabla \times \mathbf{E})$

Cross product rule:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

so

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E}$$

But $\nabla \cdot \mathbf{E} = 0$, and $\nabla \cdot \nabla = \nabla^2$, so

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

wave equation, $v = (\epsilon_0\mu_0)^{-1/2}$

Similarly, take $\partial/\partial t$ of (3), end up with

$$\nabla^2 \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

So expect EM waves to exist. Do they correspond to light?

Compare speeds: $(\epsilon_0 \mu_0)^{-1/2} = 3 \times 10^8$ m/s
Measured light speed $c = 3 \times 10^8$ m/s

Conclude light is EM wave

EM waves are *vector* waves

Previously considered scalar waves

Actually six coupled wave equations:

$(E_x, E_y, E_z, B_x, B_y, B_z)$

Components must still obey Maxwell's equations

example:

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

and

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

Can simplify for plane wave solutions:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

with $\mathbf{E}_0$, $\mathbf{B}_0$ = complex vector amplitudes

Sometimes confusing, so write out:

$$\mathbf{E}_0 = |E_{0x}|e^{i\phi_x}\hat{\mathbf{x}} + |E_{0y}|e^{i\phi_y}\hat{\mathbf{y}} + |E_{0z}|e^{i\phi_z}\hat{\mathbf{z}}$$

and actual wave is

$$\begin{aligned}\mathbf{E} =& |E_{0x}|\hat{\mathbf{x}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_x) \\ &+ |E_{0y}|\hat{\mathbf{y}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_y) \\ &+ |E_{0z}|\hat{\mathbf{z}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_z)\end{aligned}$$

Plane Waves

Have $$\begin{aligned}\frac{\partial \mathbf{E}}{\partial x} &= \frac{\partial}{\partial x}\mathbf{E}_0 e^{i(k_x x + k_y y + k_z z - \omega t)} \\ &= ik_x \mathbf{E}_0 e^{i(k_x x + k_y y + k_z z - \omega t)} \\ &= ik_x \mathbf{E}(\mathbf{r},t)\end{aligned}$$

Also $\frac{\partial \mathbf{E}}{\partial y} = ik_y\mathbf{E}$, $\frac{\partial \mathbf{E}}{\partial z} = ik_z\mathbf{E}$

and $\frac{\partial \mathbf{B}}{\partial x} = ik_x\mathbf{B}$, $\frac{\partial \mathbf{B}}{\partial y} = ik_y\mathbf{B}$, $\frac{\partial \mathbf{B}}{\partial z} = ik_z\mathbf{B}$

Thus, for plane waves, can replace

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \rightarrow i(\hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z) = i\mathbf{k}$$

Also, $\frac{\partial}{\partial t} \to -i\omega$

So wave equation becomes

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \to -k^2\mathbf{E} = -\frac{\omega^2}{c^2}\mathbf{E}$$

Solution if $k = \omega/c$, as before.

Maxwell equations become

$$\mathbf{k} \cdot \mathbf{E} = 0 \qquad \mathbf{k} \cdot \mathbf{B} = 0$$
$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B} \qquad \mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}$$

Exponentials factors $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ drop out, so

$$\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 \qquad \hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$$
$$\hat{\mathbf{k}} \times \mathbf{E}_0 = c\mathbf{B}_0 \qquad \hat{\mathbf{k}} \times \mathbf{B}_0 = -\frac{1}{c}\mathbf{E}_0$$

with $\hat{\mathbf{k}}$ = propagation direction

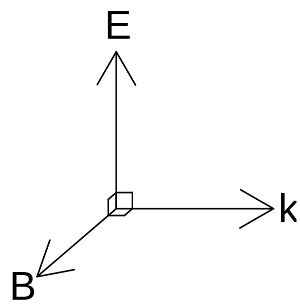
Dot products indicate $\mathbf{E}_0$, $\mathbf{B}_0 \perp \mathbf{k}$

- Say EM waves are *transverse*

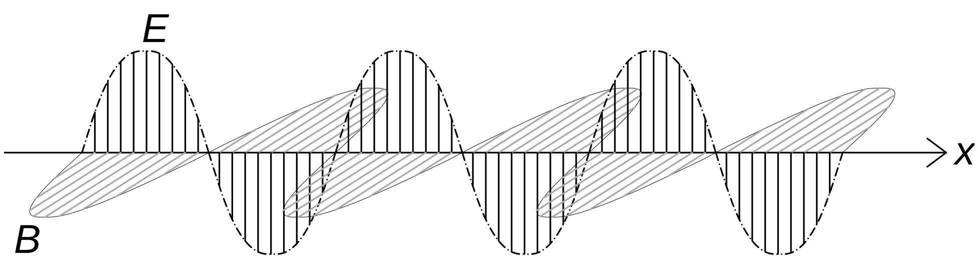
Cross products indicate $\mathbf{B}_0 = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}_0$

- So $\mathbf{B} \perp \mathbf{E}$ as well
- $|\mathbf{B}_0| = |\mathbf{E}_0|/c$

So $(\mathbf{E}, \mathbf{B}, \mathbf{k})$ form orthogonal basis



Picture: (snapshot at fixed t; $\hat{\mathbf{k}} = \hat{\mathbf{x}}$)



Question: If a wave has $\mathbf{E}$ along $\hat{\mathbf{x}}$ and $\mathbf{B}$ along $\hat{\mathbf{z}}$, what is the direction of propagation?

Question: How would my ‘snapshot’ picture evolve in time?

Frequencies (Hecht 3.6)

EM waves observed over large range of ω:

ν (Hz)	λ	Name
$< 10^5$	$>$ 3000 m	ELF wave
$10^5 - 10^9$	3000 m – 30 cm	radio wave
$10^9 - 10^{11}$	30 cm – 3 mm	microwave
$10^{11} - 10^{13}$	3 mm – 30 μm	terahertz
$10^{13} - 4 \times 10^{14}$	30 μm – 750 nm	infrared
4–8 $\times 10^{14}$	750–375 nm	visible light
$8 \times 10^{14} - 10^{16}$	375–30 nm	ultraviolet
$10^{16} - 10^{19}$	30 nm – 30 pm	x-rays
$> 10^{19}$	$<$ 30 pm	gamma rays

Optics particularly deals with "light:"
infrared to ultraviolet

Lower ν: approximations not valid
ray optics fails
wave approximations fail

Higher ν: quantum effects important
wave effects hard to see
ray optics OK, but no optical materials

Properties of plane waves

Sometimes write $\mathbf{E}_0 = E_0 \hat{\jmath}$ with $|\hat{\jmath}| = 1$

- E_0 = complex amplitude (V/m)
- $\hat{\jmath}$ = polarization vector = Jones vector

- More on polarization later

- Amplitude related to energy in wave:
  discuss now

General EM energy density (J/m^3):

$$u(\mathbf{r}) = \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2$$

Energy in volume $V = \iiint\limits_V u(\mathbf{r})\, dV$

For plane wave $|\mathbf{B}| = |\mathbf{E}|/c$. So

$$u = \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0 c^2}|\mathbf{E}|^2$$

$$= \frac{\epsilon_0}{2}|\mathbf{E}|^2 + \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \epsilon_0|\mathbf{E}|^2$$

But here $\mathbf{E}$ refers to real electric field

If $\mathbf{E} = \text{Re}\ [E_0\hat{\jmath}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$, then really

$$\begin{aligned}\mathbf{E} =& |E_{0x}|\hat{\mathbf{x}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_x)\\ &+|E_{0y}|\hat{\mathbf{y}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_y)\\ &+|E_{0z}|\hat{\mathbf{z}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_z)\end{aligned}$$

and

$$\begin{aligned}u =& |E_{0x}|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_x)\\ &+|E_{0y}|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_y)\\ &+|E_{0z}|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi_z)\end{aligned}$$

Energy oscillates in time.

For light, oscillation is rapid:
usually average over many periods

Average of $\cos^2()$ over many periods $= \frac{1}{2}$

So time-average $\quad \langle u\rangle = \frac{\epsilon_0}{2}(|E_{0x}|^2+|E_{0y}|^2+|E_{0z}|^2)$

In terms of complex fields, $\quad \langle u\rangle = \frac{\epsilon_0}{2}|\mathbf{E}|^2$

where $|\mathbf{E}|^2 = \mathbf{E}\cdot\mathbf{E}^* = |E_{0x}|^2+|E_{0y}|^2+|E_{0z}|^2$

Question: What is the *total* energy in a plane wave with amplitude E_0?

Also interested in energy flow:

Use Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$

Can show that

$$W = \iint_{\Sigma} \mathbf{S} \cdot \mathbf{d\Sigma}$$

is energy per unit time crossing surface $\boldsymbol{\Sigma}$

Units of $\mathbf{S}$ are W/m^2

Plane waves: $\mathbf{B}_0 = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}_0$, so

$$\mathbf{S} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}\,(|E_{0x}|^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_x) + |E_{0y}|^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_y) + |E_{0z}|^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_z))$$

Direction of propagation $\hat{\mathbf{k}}$
= direction of energy flow $\hat{\mathbf{S}}$

Question: What do you think $\mathbf{S}(\mathbf{r})$ looks like for a spherical wave?

Define *irradiance* (aka intensity) =
time-average of magnitude of $\mathbf{S}$

$$I = \langle|\mathbf{S}|\rangle = \frac{1}{2\mu_0 c}|E_0|^2 = \boxed{\frac{1}{2\eta_0}|\mathbf{E}_0|^2}$$

where $\eta_0 \equiv (\mu_0/\epsilon_0)^{1/2} = 377\ \Omega$
"impedance of free space"

Units check: $(\mathrm{W/m^2}) = (\mathrm{V/m})^2/\Omega$
(Recall $P = V^2/R$ from electronics)

I is most common measure of optical field strength

In terms of complex fields: $\mathbf{S} = \frac{1}{2\mu_0}\mathbf{E} \times \mathbf{B}^*$
(implicit time average!)

Poynting vector gives other properties too:

- Energy density $\langle u \rangle = \frac{|\mathbf{S}|}{c}$
- Linear momentum $\langle \mathbf{p} \rangle = \frac{1}{c^2}\iiint \mathbf{S}\, dV$
- Angular momentum $\langle \mathbf{L} \rangle = \frac{1}{c^2}\iiint \mathbf{r} \times \mathbf{S}\, dV$

Summary:

- Light is EM wave
- Coupled $\mathbf{E}$, $\mathbf{B}$ fields
- Plane waves, complex vector amplitudes
- Irradiance $I = \frac{1}{2\eta_0}|\mathbf{E}_0|^2$ (W/m^2)