Phys 531 Lecture 3 9 September 2004

Light in Matter (Hecht Ch. 3)

Last time, talked about light in vacuum:

Maxwell equations $\rightarrow$ wave equation

Light = EM wave

Today: What happens inside material?
typical example: glass
Important for understanding lenses, prisms etc.

Consider:

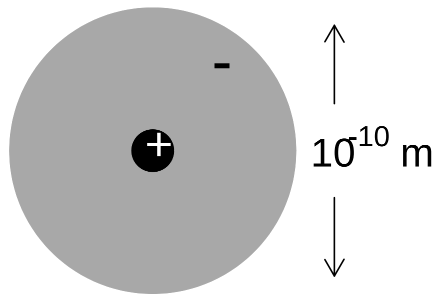
- effect on Maxwell Eqns
- index of refraction
- atomic model for index

Next time: another perspective on same question

What is matter? Collection of atoms

atom = positive nucleus
+ negative electron cloud

The atom:



More detail: use quantum mechanics
- plan to avoid here

So, matter contains charges:
can't set ρ, $\mathbf{J} = 0$ in Maxwell equations:

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

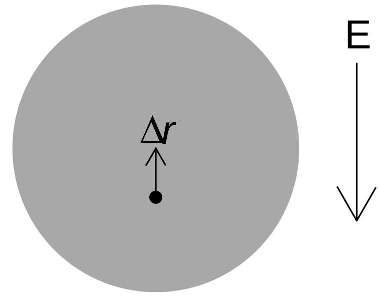
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Do we really need to know ρ and $\mathbf{J}$ exactly?
don't care about phenomena at atomic scale

On macroscopic scale, charge, current $\rightarrow 0$
Is there any macroscopic effect?

Yes, can have macroscopic *dipole moment*:

Electrons move in applied $\mathbf{E}$:



displace cloud by $\Delta\mathbf{r}$

gives atomic dipole moment $\mathbf{p} = q\Delta\mathbf{r}$

q = net charge displaced

(Expect $|\Delta r| \ll 10^{-10}$ m, $|q| \sim$ electron charge e)

Many atoms add up, give

macroscopic *polarization* $\mathbf{P}$

= net dipole moment per unit volume

If density N (atoms/m^3), then $\mathbf{P} = N\mathbf{p}$

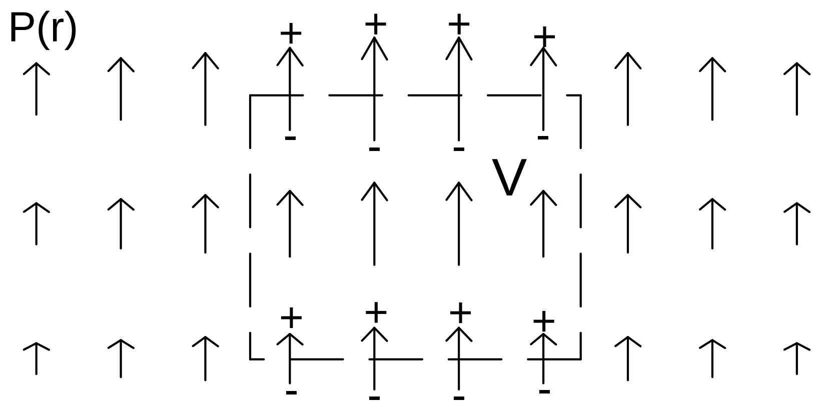
units $\mathbf{P}$ = (Cm)/m^3 = C/m^2

How does $\mathbf{P}$ come into Maxwell equations?

Must be related to ρ and $\mathbf{J}$

try to see how

Suppose $\mathbf{P}(\mathbf{r})$, test volume V:



Find net charge enclosed $Q = -\oiint \mathbf{P} \cdot \mathbf{dS}$

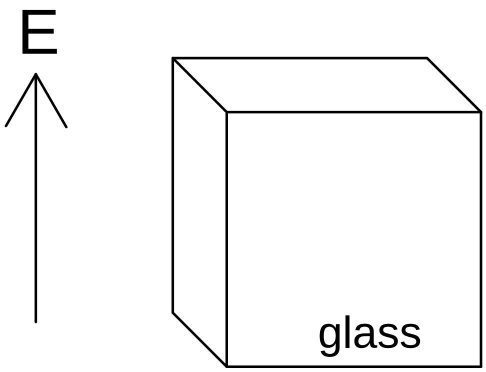
From Gauss's Theorem:

$$Q = -\iiint \nabla \cdot \mathbf{P} \, dV$$

But know $\quad Q = \iiint \rho \, dV$

Conclude $\boxed{\rho = -\nabla \cdot \mathbf{P}}$

Question: If an electric field is applied to a glass cube as shown, what is the resulting charge distribution? Does it satisfy $\rho = -\nabla \cdot \mathbf{P}$?



What happens if $\mathbf{E}$ is oscillating?

Also, changing $\mathbf{P}(t)$ gives current $\mathbf{J}$:

Charges q moving at velocity $\mathbf{v}$:
net current density $= Nq\mathbf{v}$

So

$$\mathbf{J} = Nq\mathbf{v} = Nq\frac{d\mathbf{r}}{dt} = N\frac{d\mathbf{p}}{dt}$$

$$\boxed{\mathbf{J} = \frac{d\mathbf{P}}{dt}}$$

Example: An ionized gas has a density of 10^{10} molecules/m^3 and carries an average charge of 10^{-20} C per molecule. The gas is flowing at a net speed of 100 m/s. How much charge passes through an area of 1 m^2 in a time of 1 s?

Solution:
Each 1 m^3 of gas has charge:
10^{10} molecules $\times 10^{-20}$ C/molecule $= 10^{-10}$ C.

One hundred cubes pass through test area in 1 s,
so net charge is 100×10^{-10} C $= 10^{-8}$ C.

Or:
$J = Nqv = (10^{10}\ \mathrm{m}^{-3})(10^{-20}\ \mathrm{C})(100\ \mathrm{m/s}) = 10^{-8}\ \mathrm{A/m}^2$

Put in $\mathbf{P}$, Maxwell equations become

$$\epsilon_0 \nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Still need to specify $\mathbf{P}$
Expect $\Delta \mathbf{r} \propto \mathbf{E}$, therefore $\mathbf{P} \propto \mathbf{E}$

Write: $\boxed{\mathbf{P} = \epsilon_0 \chi \mathbf{E}}$

with $\chi \equiv$ electric susceptibility
(dimensionless)

Non-linear optics:

$$P = \epsilon_0(\chi E + \chi^{(2)}E^2 + \chi^{(3)}E^3 + \ldots)$$

non-linear function

If E is small, only linear term matters

Characteristic scale: electric field from nucleus

$$E \sim \frac{e}{4\pi\epsilon_0 r^2} \approx 10^{11} \text{ V/m}$$

Corresponds to $I \sim 10^{17}$ W/m^2

We'll assume $I \ll$ this

Want more? take Phys 532 next semester

We have $\mathbf{P} = \epsilon_0\chi\mathbf{E}$

Consider wave in infinite, uniform medium

Then χ constant in space

So $\epsilon_0\nabla\cdot\mathbf{E} = -\nabla\cdot\mathbf{P}$ becomes $\nabla\cdot\mathbf{E} = -\chi\nabla\cdot\mathbf{E}$

Expect $\chi > 0$,

must have $\nabla\cdot\mathbf{E} = 0$

same as vacuum!

But also $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

becomes $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 (\chi + 1) \frac{\partial \mathbf{E}}{\partial t}$

Define electric permittivity $\epsilon = \epsilon_0 (\chi + 1)$

Then Maxwell equations become

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \epsilon \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Like vacuum, but $\epsilon_0 \to \epsilon$

Still have waves, but

speed $c \to \frac{1}{\sqrt{\epsilon \mu_0}} = \frac{c}{\sqrt{1 + \chi}}$

Define $\sqrt{1 + \chi} = n$ *index of refraction*
then $v = c/n$

Expect $\chi > 0$, so $n > 1$ and $v < c$

- Light is slower in medium than in vacuum

(Will see that's not always the case!)

Question: Since the electrons are displaced, and they have negative charge, shouldn't we normally expect $\chi < 0$?

Effect on plane waves:

Still need $k = \omega/v$

So $k = n\omega/c \equiv nk_0$

k_0 = vacuum wave number

Wave number typically increases in medium

Wave vector $\mathbf{k} = n\mathbf{k}_0$

Have $\lambda = 2\pi/k = \lambda_0/n$

λ_0 = vacuum wavelength

Wave length typically decreases in medium

Irradiance also changed:

Still $\mathbf{S} = \frac{1}{2\mu_0}\mathbf{E}_0 \times \mathbf{B}_0$

Now $\mathbf{B}_0 = \dfrac{1}{v}\widehat{\mathbf{k}} \times \mathbf{E}_0 = \dfrac{n}{c}\widehat{\mathbf{k}} \times \mathbf{E}_0$

So

$$\mathbf{S} = \frac{n}{2\mu_0 c}|\mathbf{E}_0|^2\widehat{\mathbf{k}} = \frac{n}{2\eta_0}|\mathbf{E}_0|^2\widehat{\mathbf{k}}$$

and

$$I = \frac{n}{2\eta_0}|\mathbf{E}_0|^2$$

Model for index (Hecht 3.5)

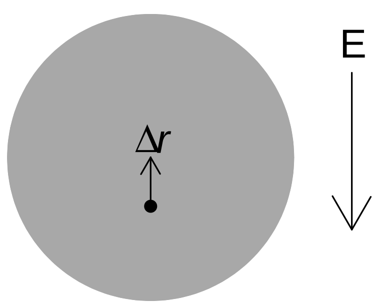
- Index of refraction is important

Could just measure for various materials, but can we relate it to a microscopic model of atom?

Quantitative accuracy: need quantum mechanics

Get basic idea with classical approach

Remember our atom model:



Displaced cloud feels linear restoring force
(for small displacements)

Total force $\mathbf{F} = q\mathbf{E} - \kappa\Delta\mathbf{r} = m\dfrac{d^2}{dt^2}\Delta\mathbf{r}$

κ = spring constant
m = mass

Expect strong response (= large n) at

$\omega \approx \omega_0 = \sqrt{\kappa/m}$

For simplicity, take $\mathbf{E}$ polarized along $\hat{\mathbf{x}}$

so $\Delta\mathbf{r} \to x$

Differential equation

$$\ddot{x} + \omega_0^2 x = \frac{q}{m} E(t)$$

Simple harmonic oscillator

infinite response at ω_0

Should also include damping force

Model with $\mathbf{F}_{\text{damp}} = -\beta \mathbf{v}$

Differential equation becomes

$$\ddot{x} + \sigma \dot{x} + \omega_0^2 x = \frac{q}{m} E(t)$$

where $\sigma = \beta/m$

Damped harmonic oscillator

Solve for plane wave $E(t) = E_0 e^{-i\omega t}$

look for solution $x(t) = x_0 e^{-i\omega t}$

find $x_0 = \dfrac{1}{\omega_0^2 - i\omega\sigma - \omega^2} \dfrac{q}{m} E_0$

or generally $\Delta\mathbf{r}(t) = \dfrac{1}{\omega_0^2 - i\omega\sigma - \omega^2} \dfrac{q}{m} \mathbf{E}(t)$

Typical resonance response

Gives dipole moment $\mathbf{p}(t) = q\Delta\mathbf{r}(t)$

and macroscopic polarization $\mathbf{P}(t) = N\mathbf{p}(t)$:

$$\mathbf{P}(t) = \frac{Nq^2}{m} \frac{1}{\omega_0^2 + i\omega\sigma - \omega^2} \mathbf{E}(t)$$

By definition $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$, so

$$\boxed{\chi = \frac{Nq^2}{\epsilon_0 m} \frac{1}{\omega_0^2 + i\omega\sigma - \omega^2}}$$

• Predicts macroscopic quantity χ in terms of microscopic quantities q, m, ω_0, σ

Note, χ is complex for $\sigma \neq 0$.

Really just our complex representation
What does it mean?

$n = \sqrt{1+\chi}$ is complex
write $\tilde{n}$ as reminder

So $\tilde{n} = n_R + i n_I$ and $\mathbf{k} = \tilde{n}\mathbf{k}_0$

Plane wave:
$$\mathbf{E} = \mathbf{E}_0 e^{i[(n_R + i n_I)\mathbf{k}_0 \cdot \mathbf{r} - \omega t]}$$
$$= \mathbf{E}_0 e^{-n_I \mathbf{k}_0 \cdot \mathbf{r}} e^{i(n_R \mathbf{k}_0 \cdot \mathbf{r} - \omega t)}$$

Really $\mathbf{E} = |E_0| \hat{\jmath} e^{-n_I \mathbf{k}_0 \cdot \mathbf{r}} \cos\left(n_R \mathbf{k}_0 \cdot \mathbf{r} - \omega t + \phi\right)$

Amplitude decays as wave propagates
models absorption
(Comes from damping in atoms)

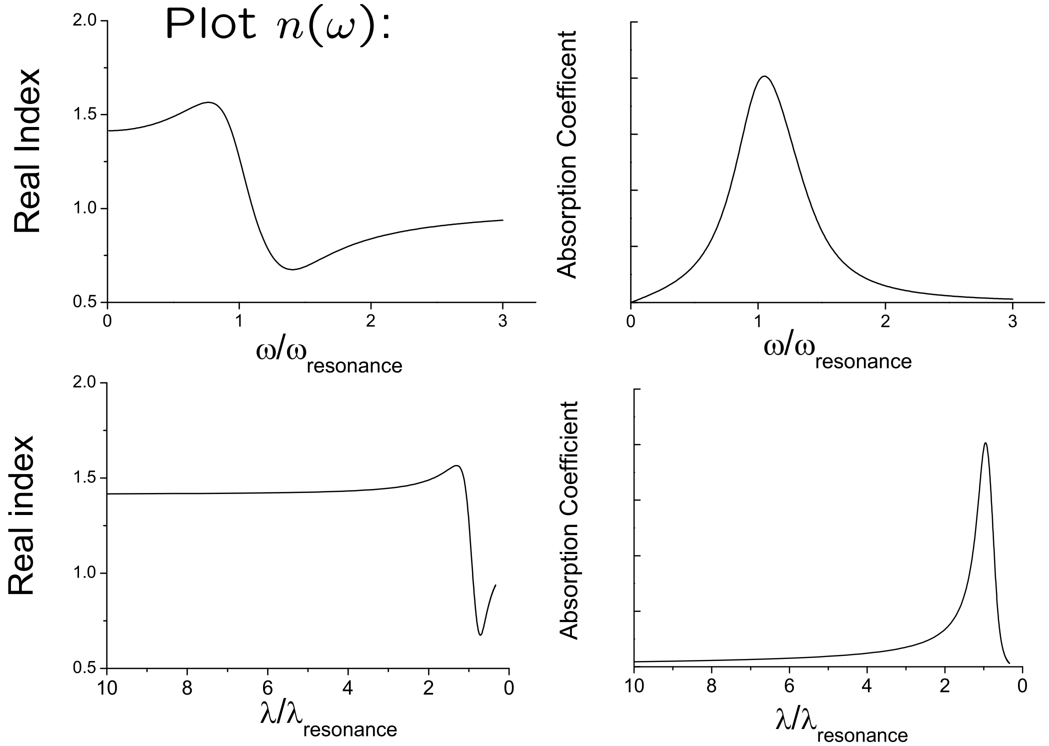
Usually write $\tilde{n} \to n + i\dfrac{\alpha}{2k_0}$
instead of $n_R + i n_I$

Say $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. Then $\mathbf{E} = \mathbf{E}_0 e^{-\alpha z/2} e^{i(n\mathbf{k}_0 z - \omega t)}$

and irradiance is $I = \dfrac{n}{2\eta_0}|E_0|^2 e^{-\alpha z}$

So α is *absorption coefficient* (units m^{-1})
I reduced by $1/e$ in distance $1/\alpha$

Plot $n(\omega)$:



Also see that n depends on ω:

Resonance at $\omega = \omega_0$

On resonance: high absorption

- bad for optics

Good materials: ω_0 lies in UV

- gives high-frequency cutoff for transmission

Below resonance $n > 1$ and $dn/d\omega > 1$

so v depends on ω

\- called *dispersion*

More on this later

Quantum mechanics gives similar result:

$$\chi(\omega) = \frac{Ne^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\sigma_j}$$

Main difference:

Many resonant frequencies ω_j
(correspond to energy transitions)

Good optical materials: no resonances in visible

Weighting factors f_j called oscillator strengths
(related to transition probabilities)

Index calculation has strange implications:

Since $n = n(\omega)$, wave velocity $v = v(\omega)$

- No longer have true wave equation
- Non-plane waves distorted in medium

Predict possible to have $n < 1$: so $v > c$?

- Meaning of v is tricky:
  still can't transmit info faster than c
- But pretty strange

Try to understand better next time

Summary

- Electric field polarizes medium, causes current flow
- EM waves in medium are similar to in vacuum, with $v = c/n$
- Medium response exhibits resonances: absorption peaks
- Glass, other good materials have no resonances in optical region