# Light in Matter: Scattering Approach

Last time, talked about light in matter:

Have charge, current terms in Maxwell eqs
Try to only consider macroscopic effects
(= polarization $\mathbf{P}$)

Result: wave equation similar to vacuum

$\epsilon_0 \to \epsilon$

$c \to c/n$

Find waves slower in medium

Do need microscopic model to calc index $n$



Today: Take a different approach

- Consider direct effect of microscopic charges on field

Find that atoms radiate new wave

total field = incident field + radiated field

Call radiated = scattered

Punch line:
Incident and scattered field travel at $v = c$,
but total field *looks* like it travels slower

Outline:

- Radiation
- Scattering by dense medium
- Scattering approach to index

Next time: Start considering boundaries between different media



## Radiation (Hecht 3.4)

Want to consider sources explicitly:
simplest source = radiating charge

(Also, nice to know where light comes from!)

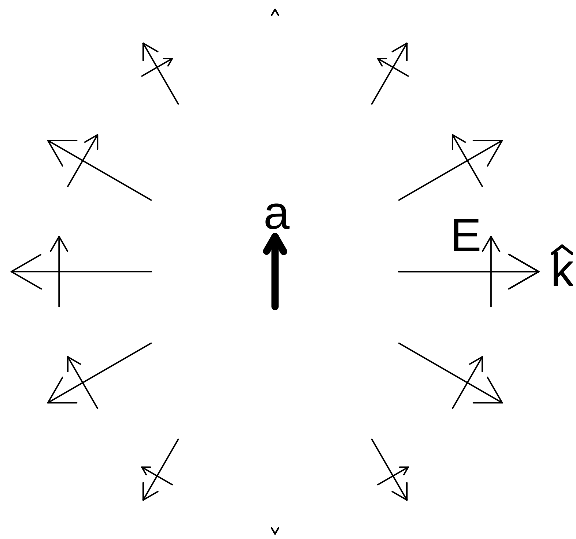
Basic result:
EM waves emitted by accelerating charge

Why? Hecht gives nice explanation
see Figure 3.28 and discussion, pg. 59

General characteristics:

- Light radiated $\perp$ acceleration $\mathbf{a}$
- $\mathbf{E}$ polarized along $\mathbf{a}$ (but $\perp \mathbf{k}$)

Picture:






More precise, solve Maxwell eqns

Simple setup:

oscillating dipole $\mathbf{p} = p_0 \hat{\mathbf{z}} \exp(-i\omega t)$

located at $\mathbf{r} = 0$

A bit hard... for derivation, see

Jackson, *Classical Electrodynamics*

Section 16.2

Result:

$$\mathbf{E} = \frac{p_0 k^3}{4\pi\epsilon_0} e^{i(kr-\omega t)} \times$$
$$\left\{ \left[ -\frac{1}{kr} - \frac{3i}{(kr)^2} + \frac{3}{(kr)^3} \right] \left( \frac{xz}{r^2}\hat{\mathbf{x}} + \frac{yz}{r^2}\hat{\mathbf{y}} - \frac{x^2+y^2}{r^2}\hat{\mathbf{z}} \right) \right.$$
$$\left. - 2\left[ \frac{i}{(kr)^2} - \frac{1}{(kr)^3} \right] \hat{\mathbf{z}} \right\}$$

$$\mathbf{B} = \frac{p_0 k^3}{4\pi\epsilon_0 c} e^{i(kr-\omega t)} \left\{ \left[ \frac{1}{kr} + \frac{i}{(kr)^2} \right] \left( \frac{y}{r}\hat{\mathbf{x}} - \frac{x}{r}\hat{\mathbf{y}} \right) \right\}$$



Real solutions are very complicated!

Optics: always assume $kr = 2\pi r/\lambda \gg 1$:

$$\mathbf{E} \to -\frac{k^3}{4\pi\epsilon_0} p_0 \frac{e^{i(kr-\omega t)}}{kr} \sin\theta\, \hat{\theta}$$

$$\mathbf{B} \to -\frac{k^3}{4\pi\epsilon_0 c} p_0 \frac{e^{i(kr-\omega t)}}{kr} \sin\theta\, \hat{\phi}$$

Spherical coords $(r, \theta, \phi)$

Called *dipole radiation field*

$\approx$ spherical wave, with extra $\sin\theta$ factor

## Scattering (Hecht 4.2)

Think about plane wave in matter

atoms $\rightarrow$ oscillating dipoles

$\rightarrow$ radiate

Try to understand effect of radiated field

I'll give more mathematical derivation:

Hecht gives more conceptual argument



Start:

Plane wave incident on single atom:

induce dipole $\mathbf{p} = \epsilon_0 \chi_1 \mathbf{E}$

with $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

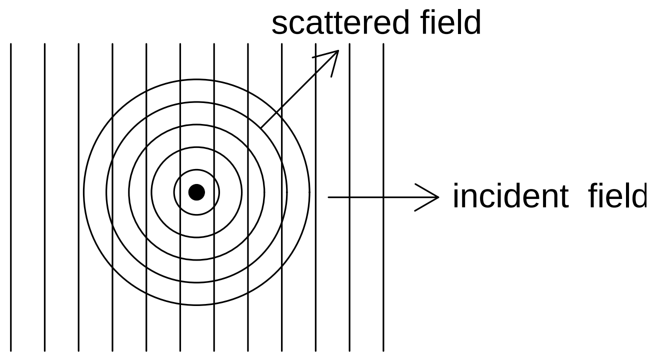
and $\chi_1 =$ "single-atom susceptability"

$= \chi / N \qquad (N = \text{density})$

Atom produces dipole field $\approx$ spherical wave

centered at atom location

Draw wave fronts:




Fields add:

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{incident}} + \mathbf{E}_{\text{scattered}}$$

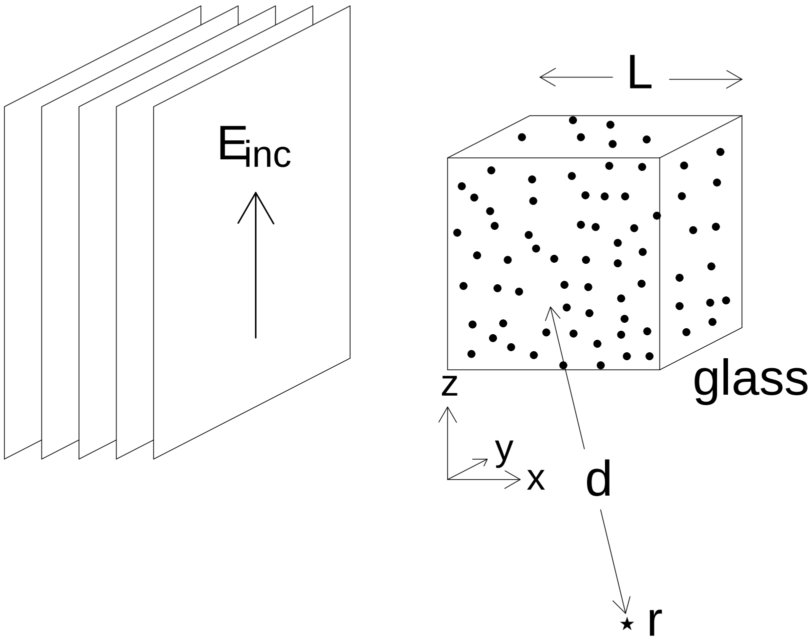


Real medium has many atoms
many $\mathbf{E}_{\text{scat}}$'s

First model: scattering by glass cube

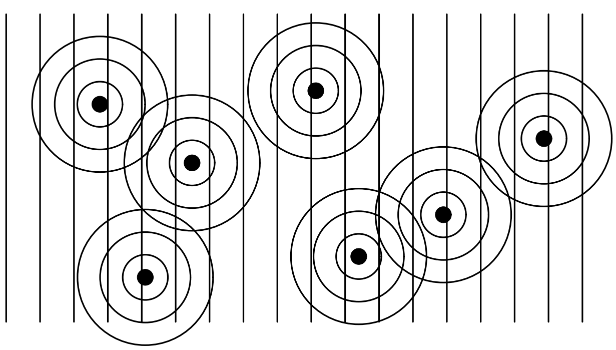
- Glass: atoms randomly distributed
- Assume cube size $L \gg \lambda$
- Measure at distance $d \gg L$
- Incident field: $\mathbf{E} = E_0 \hat{\mathbf{z}} e^{i(kx-\omega t)}$
  take $x = 0$ at front face of cube

Setup:






Need to add up scattered fields from each atom



Will see that fields tend to cancel out,
except when $\mathbf{r}$ is in *front* of medium

Write $\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{inc}}(\mathbf{r}) + \sum_j \mathbf{E}_j(\mathbf{r})$

$\mathbf{E}_j(\mathbf{r})$ = scattered field from atom $j$

Calculate $\mathbf{E}_j$:

dipole $\mathbf{p}_j = \epsilon_0 \chi_1 \mathbf{E}_{\text{inc}}(\mathbf{r}_j)$
$= \epsilon_0 \chi_1 E_0 \hat{\mathbf{z}} e^{i(kx_j - \omega t)}$

where $\mathbf{r}_j = (x_j, y_j, z_j)$ is position of atom $j$



Produces scattered field

$$\mathbf{E}(\mathbf{r}) = -\frac{k^3}{4\pi\epsilon_0} p_{j0} \frac{e^{i(kd_j - \omega t)}}{kd_j} \sin\theta_j \, \hat{\theta}_j$$

where $d_j = |\mathbf{r} - \mathbf{r}_j|$ is distance to atom $j$
Use expression for $\mathbf{p}_j$:

$$\mathbf{E}_j(\mathbf{r}) = -\frac{k^3}{4\pi\epsilon_0} \left(\epsilon_0 \chi_1 E_0 e^{ikx_j}\right) \frac{e^{i(kd_j - \omega t)}}{kd_j} \sin\theta_j \, \hat{\theta}_j$$

Use $d \gg L$ to replace
$d_j \to d$
$\theta_j \to 90^\circ$ (and $\hat{\theta}_j \to -\hat{\mathbf{z}}$)
outside of exponent

Obtain

$$\boxed{\mathbf{E}_j(\mathbf{r}) = \frac{k^2 \chi_1 E_0}{4\pi d} e^{i[k(x_j+d_j)-\omega t]}}$$

= field from atom $j$

Need to sum this over all $j$

Expect $\phi_j \equiv k(x_j + d_j)$ vary $\sim$ randomly
 equally likely + or -
 fields tend to cancel out



Show cancellation explicitly:

consider $f = \sum_j e^{i\phi_j}$

Average over possible phases:

$$\langle f \rangle = \sum_j \left\langle e^{i\phi_j} \right\rangle = \mathcal{N} \left\langle e^{i\phi} \right\rangle$$

$\mathcal{N}$ = number of atoms = $NL^3$

$$\left\langle e^{i\phi} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi}\, d\phi$$
$$= \frac{1}{2\pi i} \left( e^{i2\pi} - e^0 \right) = 0$$

So $\langle f \rangle = 0$

But cancellation not perfect

mean square field $\left\langle |f|^2 \right\rangle \neq 0$

(Remember $I \propto |E|^2 \propto |f|^2$)

Get $\left\langle |f|^2 \right\rangle = \left\langle \sum_j e^{i\phi_j} \sum_\ell e^{-i\phi_\ell} \right\rangle$

$$= \sum_{j\ell} \left\langle e^{i(\phi_j - \phi_\ell)} \right\rangle$$

If $j \neq \ell$, average is zero as before

If $j = \ell$, average = 1

So $\left\langle |f|^2 \right\rangle = \mathcal{N}$



Expect total scattered field at $\mathbf{r} \propto \mathcal{N}^{1/2}$

Means scattered field *per atom* decreases like $\mathcal{N}^{-1/2}$

In a dense medium, scattering is suppressed.

But not completely: still have $I_{\text{scat}} \propto \mathcal{N}$

seems like what you would expect!

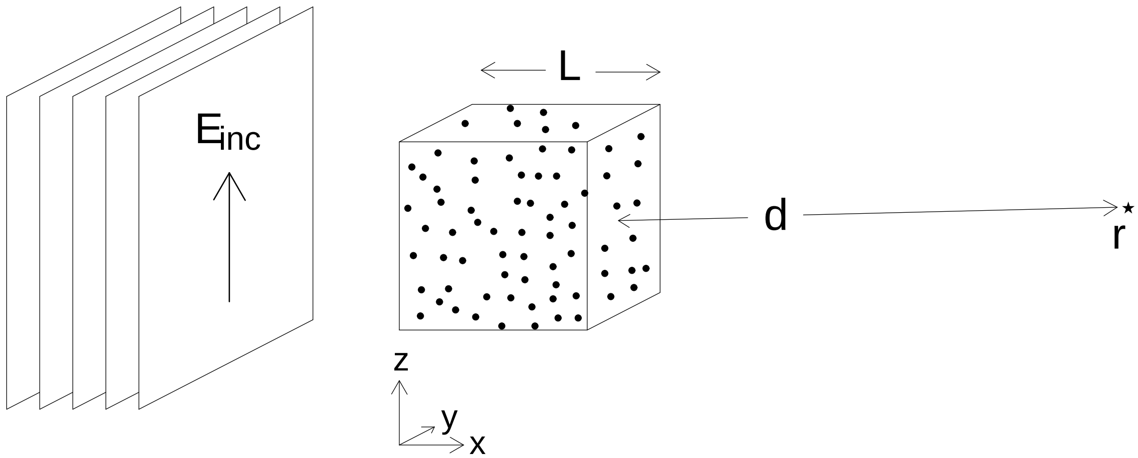
Rayleigh scattering: why sky is blue

What is scattering suppressed compared to?

- Case $L \ll \lambda^3$: phases don't cancel
- Case of forward scattering

# Forward Scattering

Consider scattered field in front of medium:




Difference: now $x_j$ and $d_j$ correlated



For $d \gg L$, have $d_j \approx d + L - x_j$

Phase $\phi_j = k(d_j + x_j) \approx k(d + L) = kx$
no longer varies

Scattered fields don't cancel out:

$$\sum_j \mathbf{E}_j(\mathbf{r}) = \sum_j \frac{k^2 \chi_1 E_0}{4\pi d} e^{i[k(x_j+d_j)-\omega t]}$$

$$= \sum_j \frac{k^2 \chi_1 E_0}{4\pi d} e^{i(kx-\omega t)}$$

$$= \mathcal{N} \frac{k^2 \chi_1}{4\pi d} E_0 e^{i(kx-\omega t)}$$

So $\mathbf{E}_{\text{scat}}$ scales as $\mathcal{N}$ and $I \propto \mathcal{N}^2$!

Forward scattering is *strong*

**Question:** We get strong forward scattering because the phases $\phi_j$ from the different atoms are all the same. This happens because the two components of the phase, $x_j$ and $d_j$, are correlated. But $x_j$ and $d_j$ are also correlated for backwards scattering... should we expect a strong backwards scattering effect?

Hint: $\phi_j = d_j + x_j$, and for backwards scattering, $d_j \approx d + x_j$



Note $\mathbf{E}_{\text{scat}}(\mathbf{r}) = \beta \mathbf{E}_{\text{inc}}(\mathbf{r})$ for

$$\beta = \frac{k^2 \chi_1 \mathcal{N}}{4\pi d}$$

forward scattered field at $\mathbf{r} \propto$ incident field at $\mathbf{r}$

Use $\chi_1 = \chi/N = \chi V/\mathcal{N}$
with $V = L^3 =$ volume of cube
Rewrite

$$\beta = \frac{k^2 \chi V}{4\pi d}$$

Conclusion:
For dense medium, forward scattering is efficient

(Also true for crystals, still get phase cancellation on sides)

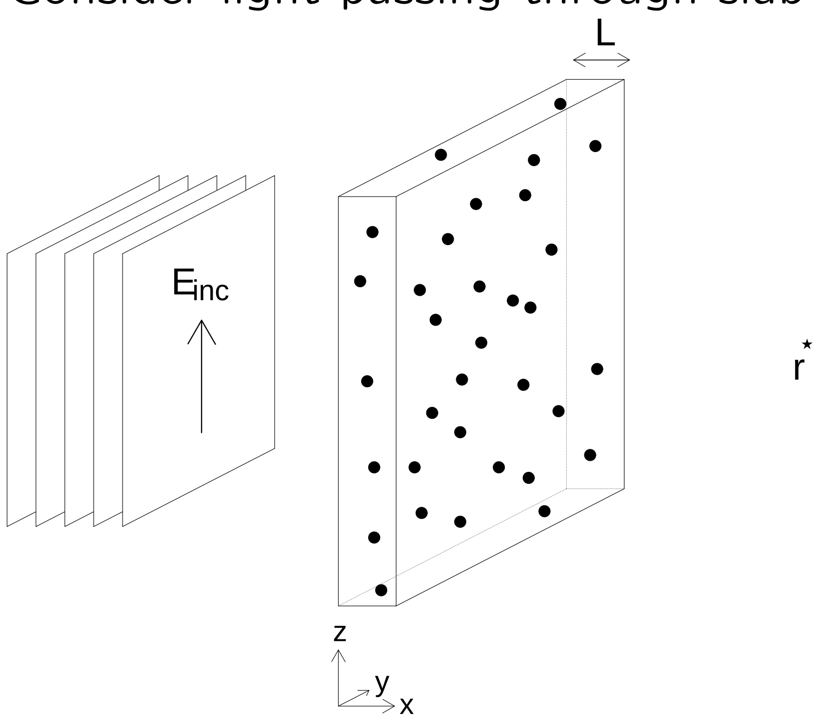
Did calculation for small cube

Result related to diffraction, more later

For now, try to apply to infinite medium

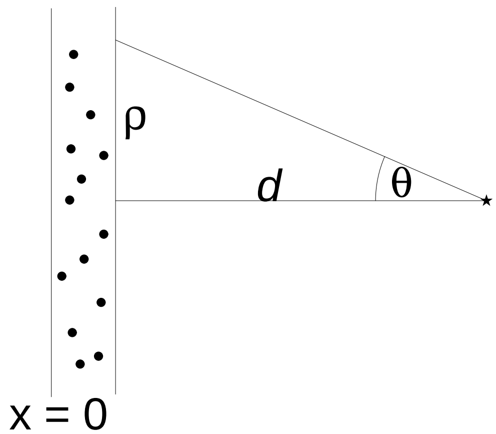


Consider light passing through slab of glass:



Slab thickness $L \ll d$
Front edge at $x = 0$
Measure at $x = d + L$

Now see forward scattering from some atoms,
not from others






Depends on angle $\theta$, $\perp$ distance $\rho$

Have $d_j = \sqrt{(d + L - x_j)^2 + \rho_j^2}$

For small $\rho_j$,

$$d_j \approx d + L - x_j + \frac{\rho_j^2}{2(d + L - x_j)}$$

$$\approx d + L - x_j + \frac{\rho_j^2}{2d}$$

(using $d \gg L, x_j$)

For forward scattering, need $\phi_j \approx$ constant, independent of $j$

$$\phi_j = k(d_j + x_j) \approx k(d + L) + \frac{k\rho_j^2}{2d}$$

Estimate $\phi_j$ can vary by $\sim$1 radian

Get forward scattering from atoms with

$$\rho_j < \rho_{\text{max}} = \sqrt{\frac{2d}{k}}$$

Defines volume $V \approx \pi\rho_{\text{max}}^2 L = \dfrac{2\pi Ld}{k}$

Atoms in $V$ contribute to forward scattering



Put this volume into formula for scattered field:

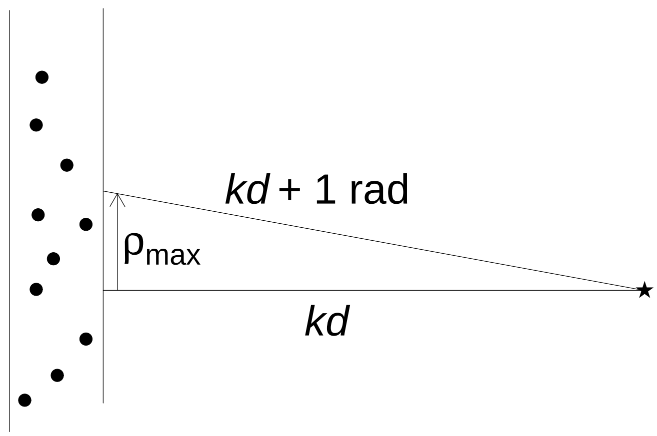
$\mathbf{E}_{\text{scat}} = \beta\mathbf{E}_{\text{inc}}$ with

$$\beta \approx \frac{k^2\chi}{4\pi d}\left(\frac{2\pi Ld}{k}\right) = \frac{kL}{2}\chi$$

**Question:** How can $d$ drop out of $\beta$? Surely, the further you are from the atoms, the weaker the scattered field should be!

Still missing one effect:
most atoms at edge of volume, have $d_j > d + L - x_j$

On average, contribute additional phase shift

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Actual calculation is a bit hard (Jackson §9.14)

Result: extra factor of $i$

$$\mathbf{E}_{\text{scat}} = i\frac{kL}{2}\chi\mathbf{E}_{\text{inc}}$$

Now go back to total field at $\mathbf{r}$:

$$\begin{aligned}\mathbf{E}_{\text{tot}}(\mathbf{r}) &= \mathbf{E}_{\text{inc}}(\mathbf{r}) + \mathbf{E}_{\text{scat}}(\mathbf{r}) \\ &= \mathbf{E}_{\text{inc}}(\mathbf{r})\left(1 + i\frac{1}{2}kL\chi\right)\end{aligned}$$

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We've implicitly assumed that scattering is weak (otherwise, scattered field would be re-scattered)

So we are limited to $kL\chi \ll 1$
say $kL$ not too big, and $\chi \ll 1$

Then free to rewrite $\mathbf{E}_{\text{tot}}$

$$\begin{aligned}\mathbf{E}_{\text{tot}}(\mathbf{r}) &= \mathbf{E}_{\text{inc}}(\mathbf{r})(1 + ikL\chi/2) \\ &= \mathbf{E}_{\text{inc}}(\mathbf{r})e^{ikL\chi/2}\end{aligned}$$

(Taylor expansion)



So have

$$\mathbf{E}_{\text{tot}} = E_0 e^{i[k(L+d)-\omega t]} e^{ikL\chi/2}$$

Rearrage exponents:

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = E_0 e^{i(kd-\omega t)} e^{ikL(1+\chi/2)}$$

For $\chi \ll 1$, recognize

$$1 + \frac{\chi}{2} \approx \sqrt{1+\chi} = n \text{ index of refraction}$$

So $\mathbf{E}_{\text{tot}}(\mathbf{r}) = E_0 e^{i(kd-\omega t)} e^{inkL}$

Final result:

$$\mathbf{E}_{\text{tot}} = E_0 e^{i[(nkL+kd)-\omega t)}$$

So $k \to nk$ for distance $L$, then $k$ for distance $d$

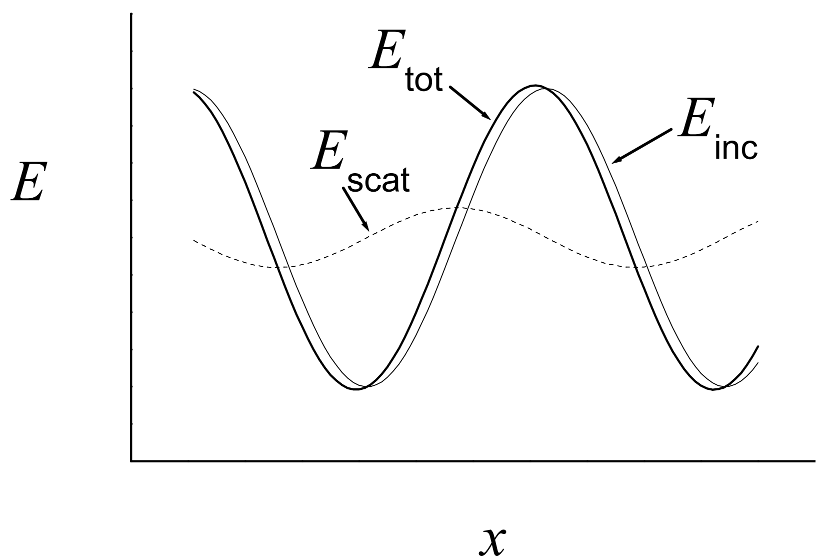
Like $k \to nk$ in medium:
Same as $v = c/n$ in medium

Same result as before!

But here all waves travel at speed $c$!?



Graphically:




Scattered field makes total field lag a bit

Upshot:

- Have "rederived" index of refraction
- Now waves travel at speed $c$ always

- Only *looks* like wave is slower in medium
  due to interference by scattered wave

Did calc for weak scattering, does generalize

So any time we see funny effect of index,
remember its really a scattering effect



**Question:**

How can we explain absorption in this scattering picture?

Hint:

If $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}$, how can we make $\mathbf{E}_{\text{inc}}$ go away?

Can explain *any* optical effect in terms of scattered fields

But usually much easier to use index approach, don't need to explicitly calculate scattered fields

Summary:

- Accelerating charges radiate

- Radiated fields from medium usually cancel, gives $I_{\text{scat}} \propto$ number of atoms

- In forward direction, radiated fields add, gives $E_{\text{scat}} \propto$ number of atoms

- Sum of incident field and forward scattered field gives total field with delayed phase: looks slow