Phys 531 Lecture 5 16 September 2004

Fermat's Principle

Last time, talked about light scattering:

total field = incident field + scattered field

Transmission through media:

- scattered field causes phase shift,
  looks like wave slows down

But individually, both waves travel at c

Today: Start considering what happens at boundary between two materials

Get Law of Reflection, Snell's Law
Generalize to Fermat's Principle

Outline:

- Optical materials
- Scattering and reflection
- Refraction
- Fermat's principle

Next time: finish studying boundaries with Fresnel relations

Optical Materials

Talked about index and absorption:
Said good materials have no resonances in visible

Be a little more specific
    What material to use in given application?

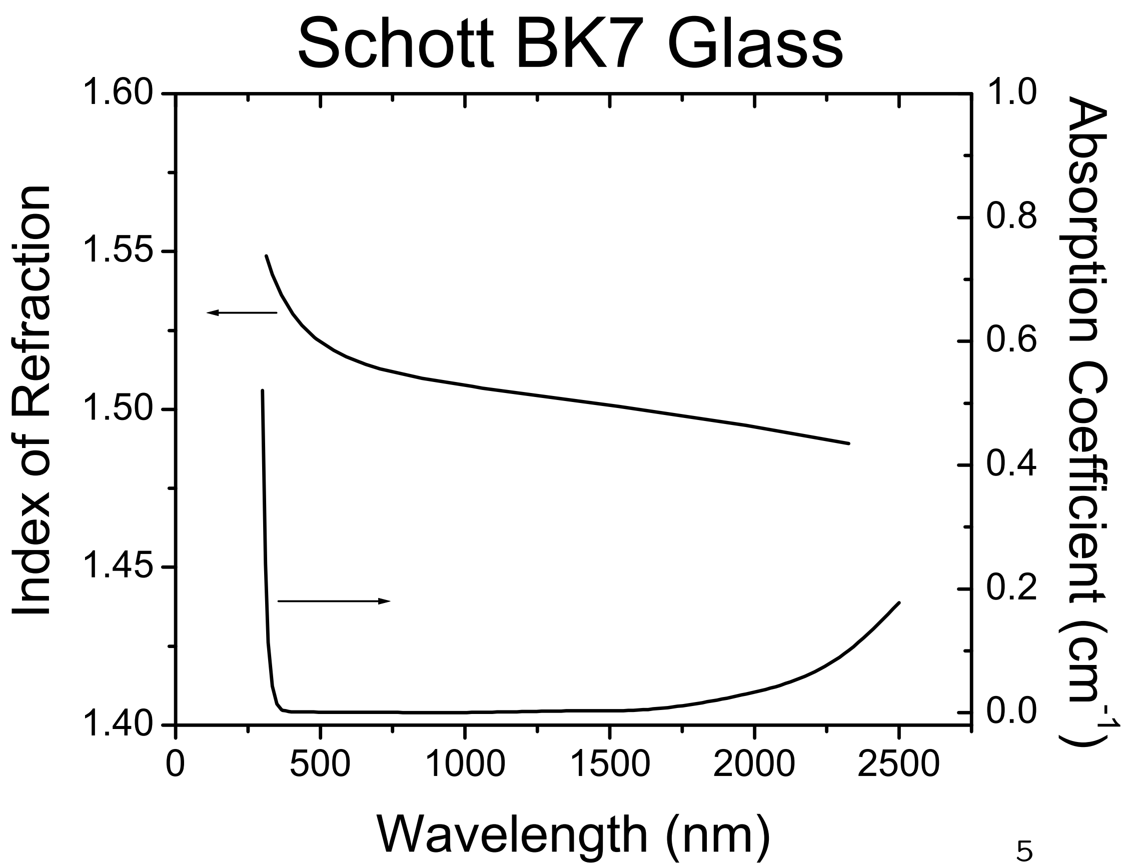
References:
- Optics catalogs (Melles Griot, CVI, Oriel)
- Schott Glass catalog

Most common optical glass: Schott BK7

SiO_2 with B_2O_5, Na_2O, CaO + others

    Index $n \approx 1.5$
    Transmission range 350 − 2000 nm
    Few bubbles or defects

Use for windows, lenses in visible, near-infrared



Resonance in UV:
electronic excitations

Resonance in IR:
molecular excitations

Impurities: several small resonances, $\lambda = 1\text{-}2\ \mu$m
don't see on graph
important for lasers, optical fibers

Question: Why does absorption increase so much more slowly in IR than in UV?

Other useful materials:

Schott SF11 glass: transmits 400 nm – 2 μm
$n \approx 1.7$

Pyrex: good mirror substrate

Suprasil: transmits 150 nm – 2 μm

MgF_2: transmits 150 nm – 6 μm
used in coatings

CaF_2: transmits 150 nm – 9 μm

Sapphire: transmits 200 nm – 6 μm

ZeSe: transmits 700 nm – 20 μm

Many others

Reflection (Hecht 4.3)

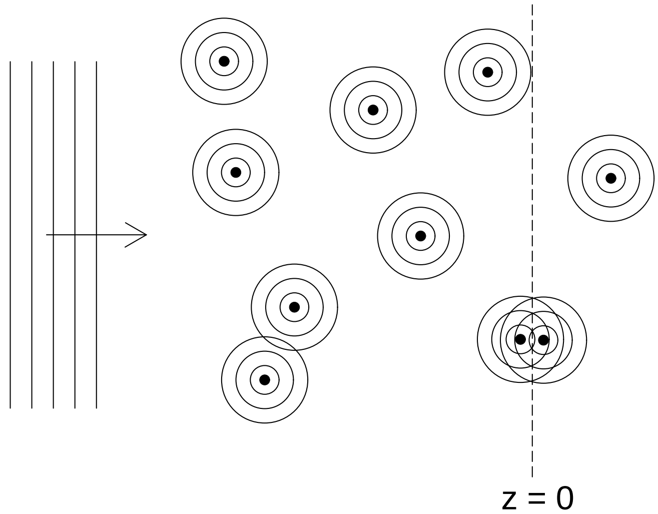
Again, two approaches possible:

- Use Maxwell with $\epsilon_0 \rightarrow \epsilon$
- Think about scattered fields

Today: take scattering approach
try to understand physics

Next time: use Maxwell, get complete answers

Start with light in piece of glass:



Why no reflection from say $z = 0$?

Scattered waves from nearby atoms cancel out

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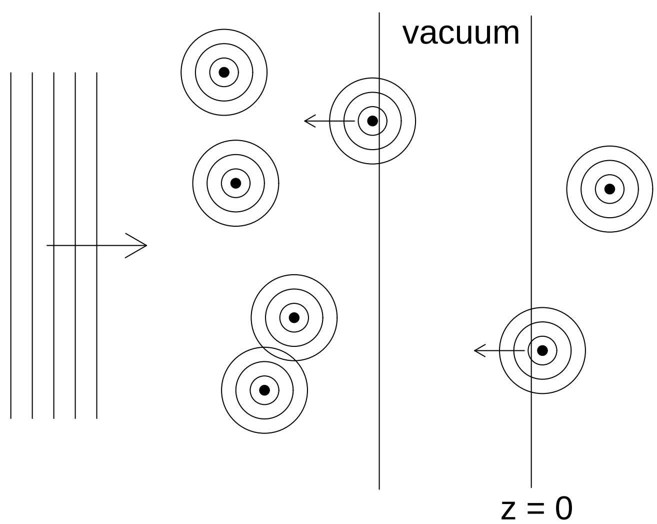
Introduce gap at $z = 0$:



Missing atoms: scattered waves don't cancel

Remove material, reflected wave appears!

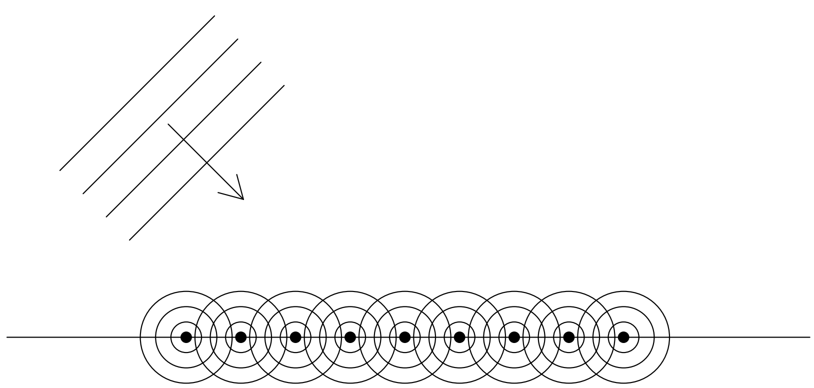
Expect wave from both surfaces
Fields should be equal and opposite:
would cancel if surfaces brought together

Funny point:
Does reflection comes from atoms at surface?

No:
Reflection = net wave from all atoms $z > 0$
(Which atoms were missing ones cancelling?)

Question: If reflected light comes from all the atoms, shouldn't the reflected field change if you introduce a second surface downstream? How can that be?

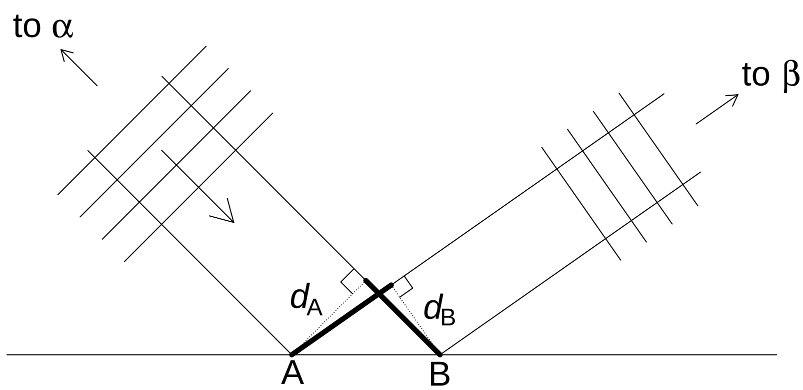
In general, incident light at an angle:



Expect reflected wave where all scattered fields have same phase

Say plane waves from distant source α,
detect at distance point β

Consider scattered field from atoms A and B



Need fields from A and B to have same phase

Say distance from α to A $= L$
distance from B to $\beta = L'$

Incident field at A $= E_0 e^{i[kL-\omega t]}$
Incident field at B $= E_0 e^{i[k(L+d_B)-\omega t]}$

Scattered field from A at β

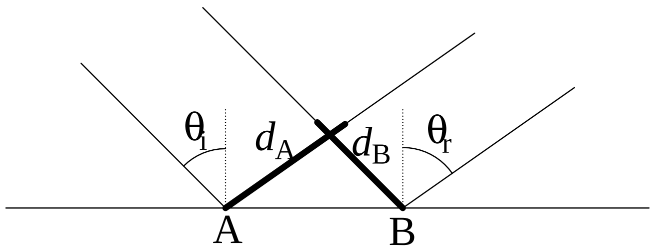
$$Ae^{ikL}e^{i[k(L'+d_A)-\omega t]}$$

Scattered field from B at β:

$$Ae^{ik(L+d_B)}e^{i[kL'-\omega t]}$$

Make phases equal: $L + L' + d_A = L + L' + d_B$

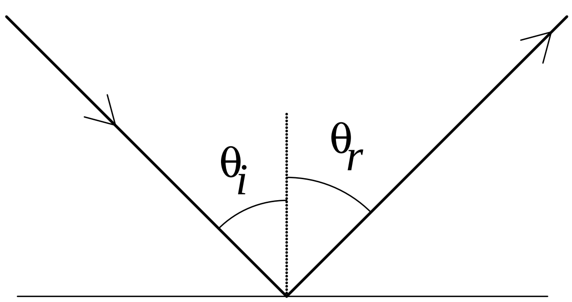
or $d_A = d_b$



angle of incidence θ_i
angle of reflection θ_r

Need $\boxed{\theta_i = \theta_r}$: Law of Reflection

Typically just draw k-vectors
= "rays"
and don't think about all this scattering stuff



Just like balls bouncing off a wall

But good to know what's going on underneath

Sometimes, need 3D version of reflection law

Define $\hat{\mathbf{u}}$ = normal to surface

Then $\hat{\mathbf{k}}_{\text{refl}}$ in plane of $\hat{\mathbf{k}}_{\text{inc}}$ and $\hat{\mathbf{u}}$

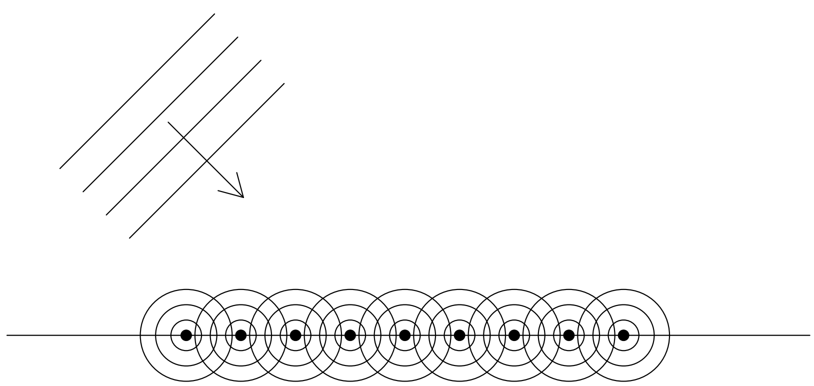
Get

$$\boxed{\hat{\mathbf{k}}_{\text{refl}} = \hat{\mathbf{k}}_{\text{inc}} - 2(\hat{\mathbf{k}}_{\text{inc}} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}}$$

Question: Should $\hat{\mathbf{u}}$ be normal pointing out of or into the surface?

Refraction (Hecht 4.4)

Think about transmitted wave



Now have $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}$

(For reflection, $\mathbf{E}_{\text{scat}}$ and $\mathbf{E}_{\text{inc}}$ more distinct)

But $\mathbf{E}_{\text{scat}}$ hard to calculate now:

- Scattered field is strong, gets rescattered
- Want field inside medium, close to charges

Be clever instead:

Incident medium: $n = n_1$
Total incident wave $= \mathbf{E}_1 e^{i(n_1 \mathbf{k}_1 \cdot \mathbf{r} - \omega t)}$
Take $\mathbf{k} = k_{1x}\hat{\mathbf{x}} + k_{1z}\hat{\mathbf{z}}$

Transmitted medium: $n = n_2$
Total transmitted wave $= \mathbf{E}_2 e^{i(n_2 \mathbf{k}_2 \cdot \mathbf{r} - \omega t)}$
Then $\mathbf{k} = k_{2x}\hat{\mathbf{x}} + k_{2z}\hat{\mathbf{z}}$

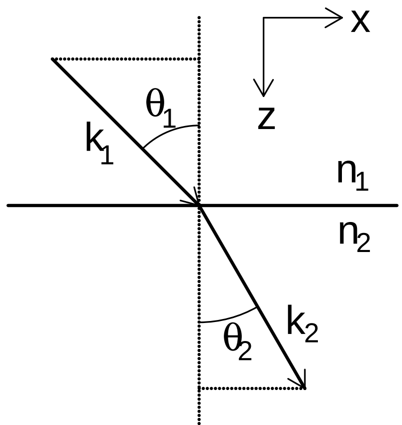
At boundary $z = 0$:
$\mathbf{E}_{\text{inc}} = \mathbf{E}_1 e^{i(n_1 k_{1x} x - \omega t)}$ $\quad\quad$ $\mathbf{E}_{\text{trans}} = \mathbf{E}_2 e^{i(n_2 k_{2x} x - \omega t)}$

Don't need $\mathbf{E}$ continous across boundary,
do need phase difference independent of x

- If boundary is uniform, how could fields be in phase at one point and out of phase at another?
- Same reason ω can't change

So need $n_1 k_{1x} = n_2 k_{2x}$

Geometry:



$k_{1x} = k_1 \sin\theta_1$
$k_{2x} = k_2 \sin\theta_2$

Here k_1 and k_2 are vacuum k's: $k_1 = k_2 = \omega/c$

So if $n_1 k_{1x} = n_2 k_{2x}$

then $\frac{n_1\omega}{c}\sin\theta_1 = \frac{n_2}{\omega}c\sin\theta_2$
or

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2} \quad \text{Snell's Law}$$

Follows from:

- $\mathbf{k} \rightarrow n\mathbf{k}$ in medium
- symmetry of surface

Generalize to 3D:

Have $\mathbf{k}_{\text{inc}}$, $\mathbf{k}_{\text{trans}}$ and surface normal $\hat{\mathbf{u}}$ in same plane

Write

$$n_1\hat{\mathbf{k}}_1 \times \hat{\mathbf{u}} = n_2\hat{\mathbf{k}}_2 \times \hat{\mathbf{u}}$$

or

$$\mathbf{k}_1 \times \hat{\mathbf{u}} = \mathbf{k}_2 \times \hat{\mathbf{u}}$$

Fermat's Principle (Hecht 4.5)

Can generalize previous results

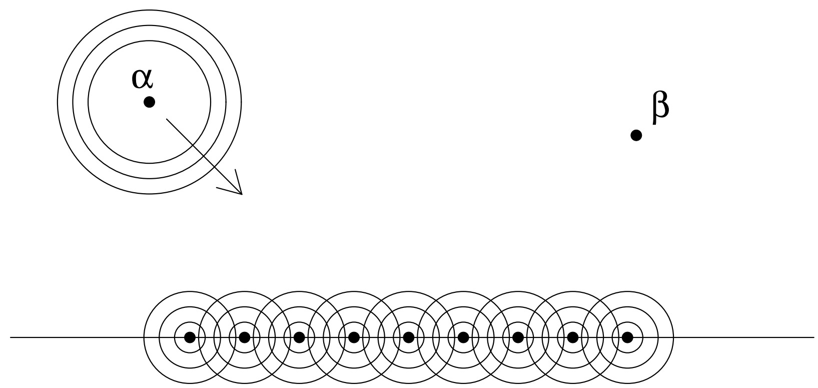
Think about reflection again

Before:

- Where does detector need to be to see reflected light?

Now ask:

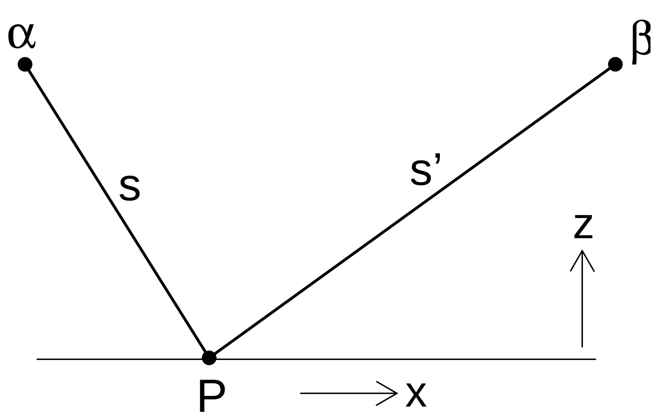
- Given source and detector positions, which points on surface contribute to detected light?



Which atoms on surface are radiating waves that interfere constructively at β?

Hard to work out geometrically, since phase of $\mathbf{E}_{\text{inc}}$ is complicated
(Even worse if surface is curved!)

Consider point P on surface



Incident field at $P = E_0 e^{i(ks-\omega t)}$

Field from P reaching β: $E_0' e^{i[k(s+s')-\omega t]}$

Write $s + s' = \mathcal{S} \equiv$ *Optical Path Length*

Net phase at β : $\phi_P = k\mathcal{S}$

Get constructive interference from points near P if fields from nearby points have same phase

Or: if $\mathcal{S}$ constant near P

If P labelled by coordinate x, want

$$\frac{d\mathcal{S}}{dx} = 0$$

Work this out

Say $\mathbf{r}_\alpha = (x_1, z_1)$ and $\mathbf{r}_\beta = (x_2, z_2)$ and $\mathbf{r}_P = (x, 0)$
(surface at $z = 0$, leave out y for now)

Then $\mathcal{S} = s + s'$

$$= \sqrt{(x - x_1)^2 + z_1^2} + \sqrt{(x - x_2)^2 + z_2^2}$$

and $\displaystyle \frac{d\mathcal{S}}{dx} = \frac{x - x_1}{\sqrt{(x - x_1)^2 + z_1^2}} + \frac{x - x_2}{\sqrt{(x - x_2)^2 + z_2^2}}$

$$= 0$$

Solve for x: $\dfrac{(x-x_1)^2}{(x-x_1)^2+z_1^2} = \dfrac{(x-x_2)^2}{(x-x_2)^2+z_2^2}$

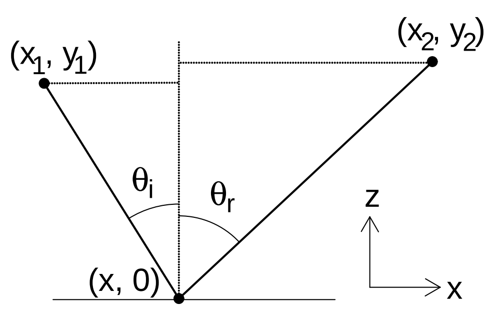
Invert: $1 + \dfrac{z_1^2}{(x-x_1)^2} = 1 + \dfrac{z_2^2}{(x-x_2)^2}$

So $\dfrac{z_1}{x-x_1} = \pm\dfrac{z_2}{x-x_2}$

From original equation, need "–" root

Solve $x = \dfrac{x_1 z_2 + x_2 z_1}{z_1 + z_2}$

Relate to geometry:



Had

$$\frac{x-x_1}{\sqrt{(x-x_1)^2+z_1^2}} = \frac{x_2-x}{\sqrt{(x-x_2)^2+z_2^2}}$$

Means $\sin\theta_i = \sin\theta_r$

Again, $\theta_i = \theta_r$, law of reflection

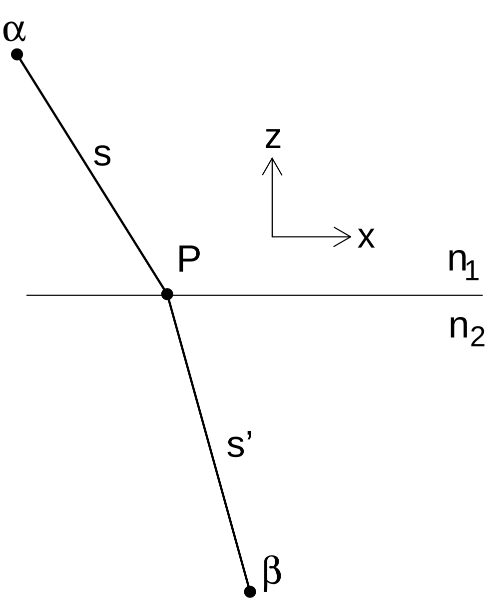
Idea: light takes path such that $\mathcal{S}$ is stationary
$\equiv$ constant for small variations in path

"Path" identifies atoms whose scattered fields add constructively

Called *Fermat's Principle*

Very powerful method

Apply to refraction



Phase from α to β

$$= n_1 ks + n_2 ks'$$

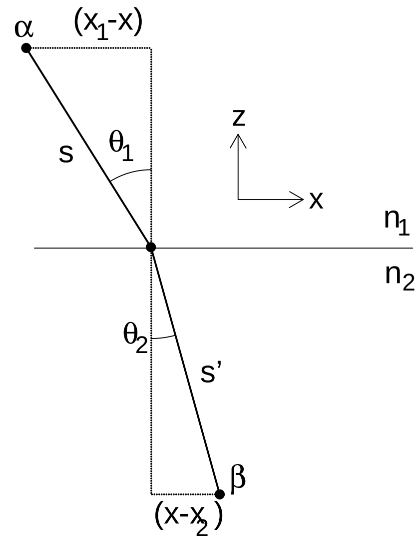
Here define $\mathcal{S} = n_1 s + n_2 s'$

Again need $d\mathcal{S}/dx = 0$

Question: Why do we want $d\mathcal{S}/dx = 0$ here?

$$\mathcal{S}(x) = n_1\sqrt{(x-x_1)^2+z_1^2} + n_2\sqrt{(x-x_2)^2+z_2^2}$$

$$\frac{d\mathcal{S}}{dx} = \frac{n_1(x-x_1)}{s} + \frac{n_2(x-x_2)}{s'} = 0$$



Again, gives
$n_1 \sin\theta_1 = n_2 \sin\theta_2$

For arbitrary path, define

$$\mathcal{S} = \int n(\mathbf{r}) ds \quad \text{(integrated along path)}$$

Usually path = sum of straight line segments
then $\int \to \sum$

Fermat's Principle: Light takes path such that $\mathcal{S}$ is *stationary*

Small variations in path, $\mathcal{S}$ doesn't change
(could be min, max, or constant)

Question: What does Fermat's principle say about light travelling through free space?

To use:

If $\mathcal{S}$ is function of parameters $\{x_i\}$, want

$$\frac{\partial \mathcal{S}}{\partial x_i} = 0 \quad \text{for all } i$$

For physics students:
Can allow arbitrary path variations
write $\delta\mathcal{S} = 0$,
get differential equation for path
Just like mechanics

Note, haven't really proven Fermat's Principle:

- Works for reflection
  makes sense from scattering picture
- Works for refraction
  scattering picture unclear
- Works in free space
  no scattering at all!

Also, ambiguous what "path" means for a wave

Revisit when we derive Huygens' Principle

Summary

For now, understand how light reflected, refracted
can understand lenses, mirrors

- Law of reflection: $\theta_i = \theta_r$
- Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$
- Fermat's principle:
  $\mathcal{S} = \int n\, ds$
  light travels path with $\mathcal{S}$ constant