Phys 531 Assignment 7 — Due October 25

1. Find the Fourier transform of the function $f(x)$ where:

$$f(x) = \begin{cases} 0 & (\text{if } x < -a/2) \\ e^{i\beta x} & (\text{if } -a/2 < x < a/2) \\ 0 & (\text{if } a/2 < x < b - a/2) \\ e^{i\beta x} & (\text{if } b - a/2 < x < b + a/2) \\ 0 & (\text{if } b + a/2 < x) \end{cases}$$

Here a and b are positive real constants with $b > a$.

2. Calculate the convolution

$$g(t) = \int_{-\infty}^{\infty} f_1(T) f_2(t - T) dT$$

for

$$f_1(t) = \begin{cases} 1 & (\text{if } -\tau_1 < t < \tau_1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$f_2(t) = \begin{cases} 1 & (\text{if } -\tau_2 < t < \tau_2) \\ 0 & (\text{otherwise}) \end{cases}$$

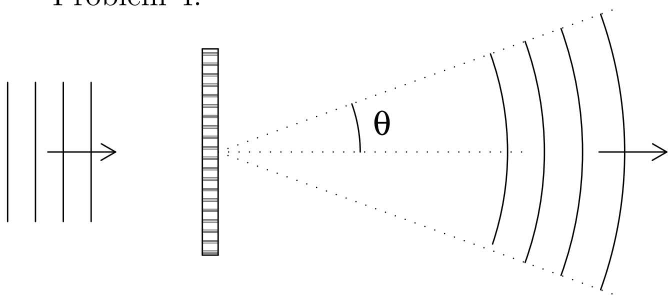
where $\tau_1 > \tau_2$. Sketch a plot of $g(t)$, and note its full-width at half-maximum.

Hint: you will need to separately consider the cases (a) $t < -(\tau_1 + \tau_2)$; (b) $-(\tau_1 + \tau_2) < t < \tau_2 - \tau_1$; (c) $\tau_2 - \tau_1 < t < \tau_1 - \tau_2$; (d) $\tau_1 - \tau_2 < t < \tau_1 + \tau_2$; (e) $\tau_1 + \tau_2 < t$. For each of these cases, it helps to draw a picture of $f_1(T)$ and $f_2(t - T)$ as functions of T.

3. For each of the following aperture functions, calculate the propagating wave $E(x, y, z)$, assuming light with wave number k:
(a) $A(x, y) = e^{iky/4}$
(b) $A(x, y) = e^{-ik(x+y)/2}$
(c) $A(x, y) = e^{ik(x+y)}$
(d) $A(x, y) = \sin^2(ky/4)$

4. Suppose that light of wavelength $\lambda = 633$ nm can be described in the $z = 0$ plane by an aperture function $A(x, y)$. It is known that $A(x, y)$ has a Fourier spectrum which is zero for $\sqrt{k_x^2 + k_y^2} > 10^6\ \text{m}^{-1}$. Show that the Fraunhofer diffraction pattern will be contained entirely within a cone and calculate the cone angle θ. (See picture on reverse.)

Problem 4:



5. The Fourier method we have developed for diffraction can be applied to other problems as well. For instance, consider a pulse of light $E(z, t)$ incident on a dispersive medium with index $n(\omega)$ and length d. The front face of the medium is at $z = 0$, and the incident pulse is

$$E(z = 0, t) \equiv A(t) = E_0 e^{-t^2/\tau^2} e^{-i\omega_0 t}$$

for pulse duration τ and carrier frequency ω_0. We can ask: what is the form of the pulse when it exits the medium?

Using the Fourier transform, the incident pulse can be written as a sum of harmonic functions $e^{-i\omega t}$. Each of these components propagates through the medium as $e^{i(kz-\omega t)}$ an thus exits after after acquiring a phase shift $\mathcal{H}(\omega) = e^{ikd}$ for $k = k(\omega) = n\omega/c$. Symbolically,

$$\text{(input) } e^{-i\omega t} \rightarrow e^{-i\omega t} e^{ikd} \text{ (output)}$$

By resumming the transmitted harmonic functions, the complete transmitted pulse can be calculated.

Suppose the wavenumber k can be approximated as

$$k(\omega_0 + \Delta) = k_0 + k_1 \Delta + \frac{1}{2} k_2 \Delta^2$$

where $\Delta = \omega - \omega_0$, $k_0 = k(\omega_0)$, $k_1 = dk/d\omega$, and $k_2 = d^2k/d\omega^2$. (Take k_0, k_1 and k_2 as givens, don't try to somehow calculate them.) Use the procedure outlined above to calculate the electric field of the transmitted pulse. In particular, find:
(a) the time at which the peak of the pulse exits the medium, and
(b) the duration of the pulse when it exits the medium (ie, the new value of τ).
Hint: The Gaussian beam example from lecture 15, slides 32–35 illustrates a similar calculation.