

Superposition and Interference

Last time: wrapped up ray optics

- aberrations
- practical considerations

Today: return to wave optics

Start developing tools for real calculations

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Outline:

- Interference
- Interference examples
- Interference in time
- Pulse propagation: group and phase v

Next time, see how to construct pulses from frequency components:

Fourier transforms

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Interference (Hecht 7.1)

Consider two waves

$$E_1 = A_1 e^{i(kz - \omega t)}$$

$$E_2 = A_2 e^{i(kz - \omega t)}$$

Note: for next six lectures, ignore polarization

If you want, assume all fields polarized along \hat{x}

Also remember really

$$E_1 = |A_1| \cos(kz - \omega t + \phi_1)$$

for $A_1 = |A_1| e^{i\phi_1}$

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Total field is sum of individual waves

$$E_{\text{tot}} = E_1 + E_2 = (A_1 + A_2) e^{i(kz - \omega t)}$$

What is the real field?

Need magnitude and phase of $A_{\text{tot}} = A_1 + A_2$

Work out:

$$\begin{aligned} |A_{\text{tot}}|^2 &= A_{\text{tot}} \cdot A_{\text{tot}}^* \\ &= (|A_1| e^{i\phi_1} + |A_2| e^{i\phi_2}) (|A_1| e^{-i\phi_1} + |A_2| e^{-i\phi_2}) \\ &= |A_1|^2 + |A_2|^2 + |A_1| |A_2| e^{i(\phi_1 - \phi_2)} \\ &\quad + |A_1| |A_2| e^{-i(\phi_1 - \phi_2)} \\ &= |A_1|^2 + |A_2|^2 + 2|A_1| |A_2| \cos(\phi_1 - \phi_2) \end{aligned}$$

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and

$$\begin{aligned}\tan \phi_{\text{tot}} &= \frac{\text{Im } A_{\text{tot}}}{\text{Re } A_{\text{tot}}} \\ &= \frac{\text{Im } A_1 + \text{Im } A_2}{\text{Re } A_1 + \text{Re } A_2} \\ &= \frac{|A_1| \sin \phi_1 + |A_2| \sin \phi_2}{|A_1| \cos \phi_1 + |A_2| \cos \phi_2}\end{aligned}$$

So, can find amplitude and phase of total wave

Usually amplitude is more interesting:

$$|A_{\text{tot}}|^2 \propto I_{\text{tot}}$$

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$$|A_{\text{tot}}|^2 = |A_1|^2 + |A_2|^2 + 2|A_1||A_2| \cos(\phi_1 - \phi_2)$$

Lets us be quantitative about interference:

“Constructive” interference

when $\phi_1 \approx \phi_2$

$$|A_{\text{tot}}|^2 \approx (|A_1| + |A_2|)^2$$

“Destructive” interference

when $\phi_1 \approx -\phi_2$

$$|A_{\text{tot}}|^2 \approx (|A_1| - |A_2|)^2$$

Given any $\delta \equiv \phi_1 - \phi_2$, we have formula

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Note if $\delta = \pi/2$ then

$$|A_{\text{tot}}|^2 = |A_1|^2 + |A_2|^2$$

Irradiances add

Irradiances also add if δ fluctuates randomly

$$I_{\text{tot}} = \frac{1}{2\eta_0} \langle |E|^2 \rangle \propto \langle |A_{\text{tot}}|^2 \rangle$$

so interference term $\rightarrow 0$ if $\langle \cos \delta \rangle = 0$

Then say waves are *incoherent*

Question: Can two harmonic waves be incoherent?

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Generalizes to N waves:

$$A_{\text{tot}} = \sum_n A_n$$

$$\begin{aligned} |A_{\text{tot}}|^2 &= \sum_n A_n \sum_m A_m^* \\ &= \sum_{n,m} |A_n| |A_m| e^{i(\phi_n - \phi_m)} \\ &= \sum_{n=1}^N |A_n|^2 + \sum_{n \neq m} |A_n| |A_m| e^{i(\phi_n - \phi_m)} \end{aligned}$$

Second term $\rightarrow 0$ for incoherent fields

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Also get

$$\tan \phi_{\text{tot}} = \frac{\sum_n |A_n| \sin \phi_n}{\sum_n |A_n| \cos \phi_n}$$

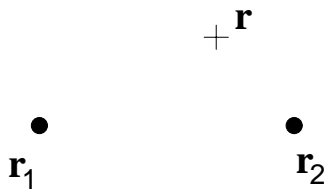
For incoherent fields, ϕ_{tot} fluctuates randomly

More about incoherent fields at end of course
generally assume coherent now

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Look closer at phases

Say waves generated by two point sources



Field from 1 = $A_1 e^{i(k|\mathbf{r}-\mathbf{r}_1|-\omega t)}$

Field from 2 = $A_2 e^{i(k|\mathbf{r}-\mathbf{r}_2|-\omega t)}$

Total field $|E_{\text{tot}}|^2 = |E_1 + E_2|^2 = |A_1|^2 + |A_2|^2$
 $+ 2|A_1||A_2| \cos(\phi_1 - \phi_2 + k|\mathbf{r} - \mathbf{r}_1| - k|\mathbf{r} - \mathbf{r}_2|)$

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Interference term depends on position:
 get *interference pattern* in space

In medium, $k = nk_o$

Optical path length difference $\Lambda = \mathcal{S}_1 - \mathcal{S}_2$

$$\Lambda = n(|\mathbf{r}_1 - \mathbf{r}| - |\mathbf{r}_2 - \mathbf{r}|)$$

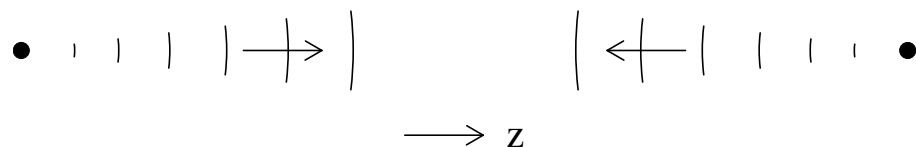
Write interference phase $\delta = \phi_1 - \phi_2 + k_0\Lambda(\mathbf{r})$

Like in scattering picture:

\mathcal{S} determines interference effects

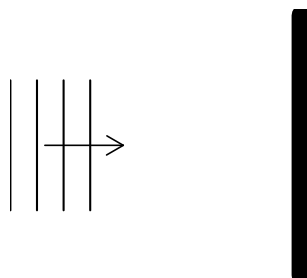
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Example: Two sources at ∞ (Hecht 7.1.4)



Then fields \rightarrow plane waves

Equivalent to plane wave reflected by mirror



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What is E_{tot} ?

$$E_{\text{tot}} = A_1 e^{i(kz - \omega t)} + A_2 e^{i(-kz - \omega t)}$$

Suppose $A_1 = A_2$

Then

$$\begin{aligned} E_{\text{tot}} &= A_1 e^{-i\omega t} (e^{ikz} + e^{-ikz}) \\ &= 2A_1 e^{-i\omega t} \cos(kz) \end{aligned}$$

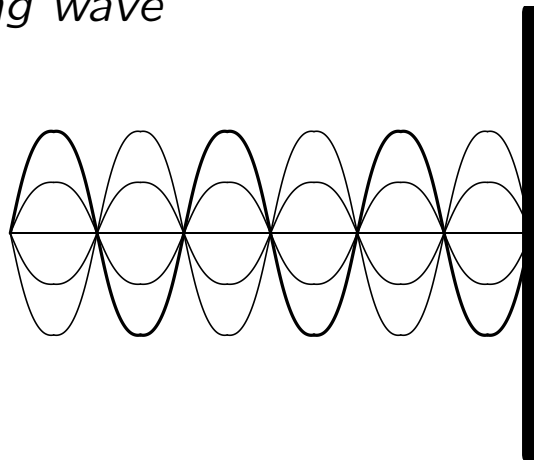
So $|E_{\text{tot}}| = 2|A_1| |\cos kz|$

and real field $E_{\text{tot}} = 2|A_1| \cos(\phi_1 - \omega t) \cos(kz)$

Not a travelling wave!

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Called *standing wave*



Fixed pattern oscillating in time

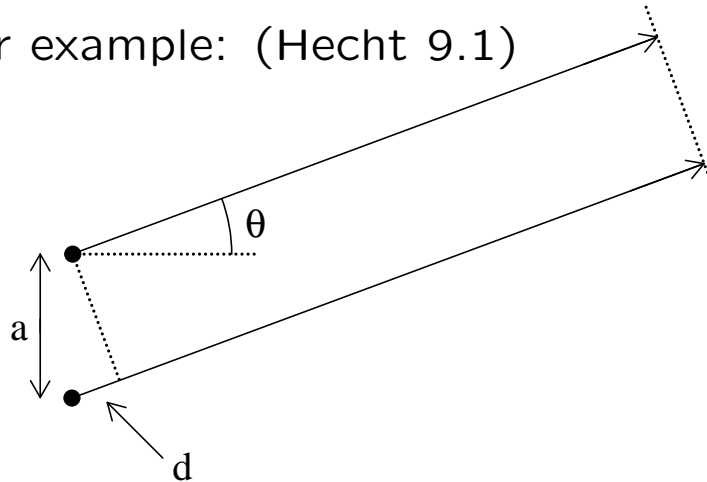
Nodes: points where $E_{\text{tot}} = 0$

Antinodes: points where $|E_{\text{tot}}| = \max = 2|A_1|$

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Standing wave is one interference pattern

Another example: (Hecht 9.1)



Two point sources separated by a

Observation point $\rightarrow \infty$ at angle θ

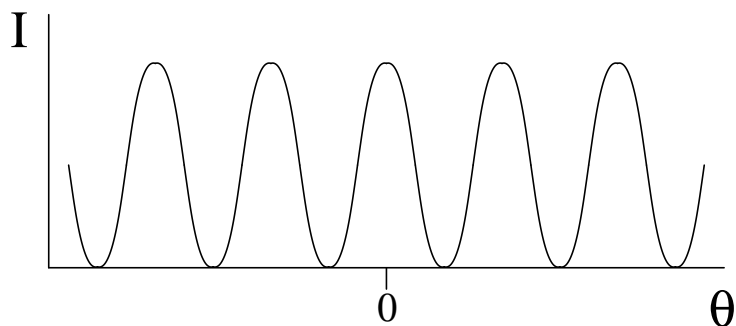
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Here $\Lambda = d = a \sin \theta \approx a\theta$

If $A_1 = A_2$, then

$$\begin{aligned} |E_{\text{tot}}|^2 &= |A_1|^2 + |A_2|^2 + 2|A_1||A_2| \cos(\phi_1 - \phi_2 + k_0\Lambda) \\ &= 2|A_1|^2(1 + \cos k_0\Lambda) \\ &= 4|A_1| \cos^2 \left(\frac{k_0\Lambda}{2} \right) \\ &\approx 4|A_1| \cos^2 \left(\frac{k_0 a \theta}{2} \right) \end{aligned}$$

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Nodes where $k_0 \Lambda / 2 = (n + \frac{1}{2})\pi$

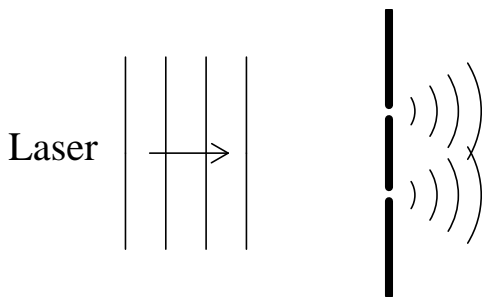
$$k_0 a \theta = \left(n + \frac{1}{2}\right) 2\pi$$

or $\theta \approx \frac{\lambda}{a} \left(n + \frac{1}{2}\right)$ since $k_0 = \frac{2\pi}{\lambda}$

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Demo: two slit interference

Setup:



Due to diffraction, slits \approx point sources

For large θ , more complicated:

Can't use $\sin \theta \approx \theta$

Can't treat slit as point source

For now, gives idea

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Basic points:

- Calculate field at position \mathbf{r} by algebraically summing incident waves
- Phase of each field depends on
 - source phase
 - optical path length between source and \mathbf{r}

Question: If the phase of the two point sources were different, how would the two-slit pattern change?

Question: What if the amplitudes of the sources were different?

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Temporal Interference (Hecht 7.2)

Can also get interference in time

Say $E_1 = A_1 e^{i(k_1 z - \omega_1 t)}$

$$E_2 = A_2 e^{i(k_2 z - \omega_2 t)}$$

with $k_i = \frac{n(\omega_i)\omega_i}{c}$ and $\omega_1 \neq \omega_2$

What is total field?

$$E_{\text{tot}} = E_1 + E_2$$

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Define $\alpha_i = k_i z - \omega_i t + \phi_i$

$$\begin{aligned} |E_{\text{tot}}|^2 &= (|A_1|e^{i\alpha_1} + |A_2|e^{i\alpha_2}) (|A_1|e^{-i\alpha_1} + |A_2|e^{-i\alpha_2}) \\ &= |A_1|^2 + |A_2|^2 + 2|A_1||A_2|\cos(\alpha_1 - \alpha_2) \end{aligned}$$

Interference term =

$$\cos \left[(k_1 - k_2)z + (\omega_1 - \omega_2)t + \phi_1 - \phi_2 \right]$$

At fixed z , irradiance oscillates at $\omega_1 - \omega_2$
 \equiv *beat note*

Identical effect with sound waves – demo

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Often useful to think of E_{tot} as single plane wave
with changing amplitude

Say $|A_1| = |A_2|$:

$$E_{\text{tot}} = |A_1|e^{i(k_1 z - \omega_1 t + \phi_1)} + |A_1|e^{i(k_2 z - \omega_2 t + \phi_2)}$$

Define $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ and $\omega_m = \omega_1 - \omega_2$

$$\begin{aligned} \text{Then } \omega_1 &= \bar{\omega} + \frac{\omega_m}{2} \\ \omega_2 &= \bar{\omega} - \frac{\omega_m}{2} \end{aligned}$$

Define similarly for \bar{k} , k_m , $\bar{\phi}$, ϕ_m

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Then write

$$\begin{aligned}
 E_{\text{tot}} &= |A_1| e^{i(\bar{k}z - \bar{\omega}t + \bar{\phi})} \\
 &\times \left[e^{\frac{i}{2}(k_m z - \omega_m t + \phi_m)} + e^{-\frac{i}{2}(k_m z - \omega_m t + \phi_m)} \right] \\
 &= 2|A_1| e^{i(\bar{k}z - \bar{\omega}t + \bar{\phi})} \cos\left(\frac{k_m z - \omega_m t + \phi_m}{2}\right)
 \end{aligned}$$

Looks like plane wave $e^{i(\bar{k}z - \bar{\omega}t)}$ with amplitude

$$A_{\text{tot}}(z, t) = 2|A_1| e^{i\bar{\phi}} \cos\left(\frac{k_m z - \omega_m t + \phi_m}{2}\right)$$

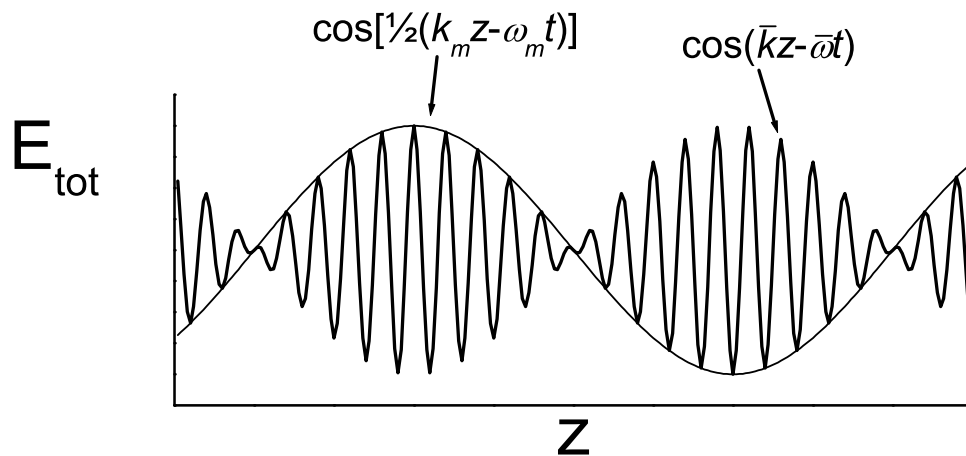
Most useful if $\omega_m \ll \bar{\omega}$

Then A_{tot} slowly varying

Field \approx plane wave at any z, t

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Picture (at fixed t):



This is example of a *modulated* wave
like a plane wave,
but amplitude and/or phase slowly varies

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Call $e^{i(\bar{k}z - \bar{\omega}t)}$ = carrier wave

Call $A_{\text{tot}}(z, t)$ = envelope function
“contains” carrier wave

Might ask, how fast does wave propagate?

Important: envelope used to transmit information

Define *phase velocity*

= speed of carrier wave

= speed of individual peaks

$$v_{\text{phase}} = \frac{\bar{\omega}}{\bar{k}} = c \frac{\omega_1 + \omega_2}{n_1\omega_1 + n_2\omega_2}$$

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If $\omega_1 \approx \omega_2$ then $n_1 \approx n_2$
and $v_{\text{phase}} \rightarrow c/n$

Makes sense

Define *group velocity* = speed of envelope

$$v_{\text{group}} = \frac{\omega_m}{k_m} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = c \frac{\omega_1 - \omega_2}{n_1\omega_1 - n_2\omega_2}$$

If $\omega_m \ll \bar{\omega}$, then get

$$v_{\text{group}} \rightarrow \left. \frac{d\omega}{dk} \right|_{\bar{\omega}} \neq v_{\text{phase}}$$

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So envelope travels at entirely different speed

Generalizes to any modulated wave:

Complicated envelopes =

more frequency components

⇒ single pulse of light travels at v_{group}

Usually easiest to evaluate

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega} \left(\frac{n\omega}{c} \right) = \frac{1}{c} \left(\frac{dn}{d\omega} \bar{\omega} + n \right)$$

$$\text{So } v_g = \frac{c}{n + \bar{\omega} \frac{dn}{d\omega}}$$

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In vacuum, $n = 1$ and $dn/d\omega = 0$:

$$\text{SO } v_{\text{group}} = v_{\text{phase}} = c$$

Only matters in medium

Recall for normal dispersion, $dn/d\omega > 0$

$$\text{Then } v_{\text{group}} < \frac{c}{n} = v_{\text{phase}}$$

Example: In BK7 glass, $n(\lambda = 546 \text{ nm}) = 1.51872$, and $n(588 \text{ nm}) = 1.5168$. What is the difference between the group and phase velocities near these wavelengths?

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Solution:

Average $n = 1.5178$, so phase velocity $\approx 0.6588c$.

For v_g , need $\omega \frac{dn}{d\omega} = \omega \frac{dn}{d\lambda} \frac{d\lambda}{d\omega}$

$$\text{Have } \frac{dn}{d\lambda} = \frac{1.51872 - 1.5168}{546 - 588} = -4.6 \times 10^{-5} \text{ nm}^{-1}$$

Also $\lambda = \frac{2\pi c}{\omega}$, so $\omega \frac{d\lambda}{d\omega} = -\omega \frac{2\pi c}{\omega^2} = -\lambda$. Use $\bar{\lambda} = 567 \text{ nm}$

$$\text{So } \omega \frac{dn}{d\omega} = -\lambda \frac{dn}{d\lambda} = 576 \text{ nm} \times 4.6 \times 10^{-5} \text{ nm}^{-1} = 0.0265$$

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} = \frac{c}{1.5178 + 0.0265} = 0.6475c$$

$$\text{Difference } \boxed{v_p - v_g = 0.011c \approx 2\%}$$

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Usually a small effect

But not always! Near resonance, $|\omega \frac{dn}{d\omega}| \gg 1$

Generally, v_{phase} and v_{group} both arbitrary

Can get $n < 1$ near resonance $\Rightarrow v_p > c$

Get $v_g > c$ for large anomalous dispersion

Didn't worry about $v_p > c$:

carrier has no information

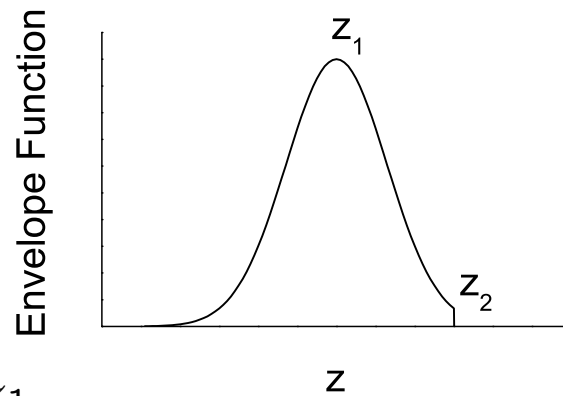
Should we worry about $v_g > c$?

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Remember, in scattering picture fields travel at c
→ c remains ultimate speed limit

What does $v_g > c$ mean?

Consider pulse of light:



Two features:

- Peak at some position z_1
- Leading edge at position z_2

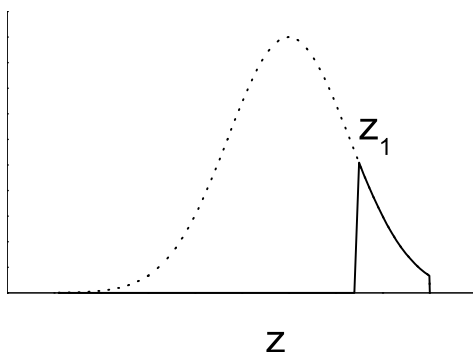
Leading edge indicates time you initiated pulse:
physical discontinuity

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Group velocity gives speed of peak
Can be larger than c

Simple example:

Suppose medium transmits for time $\ll \Delta t_{\text{pulse}}$
Only front part of pulse transmitted



Peak z_1 has advanced:
 $v_g > c$

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However, can show that leading edge *always* travels at c

Why? Property of discontinuity

No definite carrier $\bar{\omega}$, and ω_m not small
simple analysis fails

More: see Jackson §7.11

Question: Can there really be a discontinuity in the electric field? Aren't Maxwell's equations continuous?

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Summary

- Compute total field by summing components
- Result generally depends on position: interference pattern
- If waves have different ω , interference depends on time
- Pulse of light travels at v_{group}
- Information always travels $v \leq c$

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