Phys $531 \quad$ Lecture 3
Light in Matter (Hecht Ch. 3)

Last time, talked about light in vacuum:
Maxwell equations $\rightarrow$ wave equation
Light $=$ EM wave

Today: What happens inside material? typical example: glass
Important for understanding lenses, prisms etc.

Consider:

- effect on Maxwell Eqns
- index of refraction
- atomic model for index

Next time: another perspective on same question

What is matter? Collection of atoms
atom $=$ positive nucleus

+ negative electron cloud

The atom:


More detail: use quantum mechanics

- plan to avoid here

So, matter contains charges:
can't set $\rho, \mathbf{J}=0$ in Maxwell equations:

$$
\begin{array}{ll}
\epsilon_{0} \nabla \cdot \mathbf{E}=\rho & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B}=\mu_{0} \mathbf{J}+\epsilon_{0} \mu_{0} \frac{\partial \mathbf{E}}{\partial t}
\end{array}
$$

Do we really need to know $\rho$ and $\mathbf{J}$ exactly? don't care about phenomena at atomic scale

On macroscopic scale, charge, current $\rightarrow 0$ Is there any macroscopic effect?

Yes, can have macroscopic dipole moment:
Electrons move in applied E:
(Nuclei fixed)

displace cloud by $\Delta \mathbf{r}$
gives atomic dipole moment $p=q \Delta r$
$q=$ net charge displaced
(Expect $|\Delta r| \ll 10^{-10} \mathrm{~m},|q| \sim$ electron charge $e$ )

Many atoms add up, give macroscopic polarization $\mathbf{P}$
$=$ net dipole moment per unit volume
If density $N$ (atoms $/ \mathrm{m}^{3}$ ), then $\mathbf{P}=N \mathbf{p}$ units $P=(C m) / m^{3}=C / m^{2}$

How does $\mathbf{P}$ come into Maxwell equations?
Must be related to $\rho$ and $\mathbf{J}$
try to see how

Suppose $\mathbf{P}(\mathbf{r})$, test volume $V$ :


Claim net charge enclosed $Q=-\oiiint \mathbf{P} \cdot \mathbf{d S}$

From Gauss's Theorem:

$$
Q=-\iiint \nabla \cdot \mathbf{P} d V
$$

But know $\quad Q=\iiint \rho d V$
Conclude $\rho=-\nabla \cdot \mathbf{P}$

Question: If a uniform electric field is applied to a glass cube as shown, what is the resulting charge distribution? Explain how it satisfies $\rho=-\nabla \cdot \mathbf{P}$.


What happens if $\mathbf{E}$ is oscillating?

Also, changing $\mathbf{P}(t)$ gives current $\mathbf{J}$ :
Charges $q$ moving at velocity v : net current density $=N q \mathbf{v}$

So

$$
\begin{aligned}
& \mathbf{J}=N q \mathbf{v}=N q \frac{d \mathbf{r}}{d t}=N \frac{d \mathbf{p}}{d t} \\
& \mathbf{J}=\frac{d \mathbf{P}}{d t}
\end{aligned}
$$

## Example: Understanding J

An ionized gas has a density of $10^{10}$ molecules $/ \mathrm{m}^{3}$ and carries an average charge of $10^{-20} \mathrm{C}$ per molecule. The gas is flowing at a net speed of $100 \mathrm{~m} / \mathrm{s}$. How much charge passes through an area of $1 \mathrm{~m}^{2}$ in a time of 1 s ?

## Solution:

Each $1 \mathrm{~m}^{3}$ of gas has charge:
$10^{10}$ molecules $\times 10^{-20} \mathrm{C} /$ molecule $=10^{-10} \mathrm{C}$.
One hundred cubes pass through test area in 1 s , so net charge is $100 \times 10^{-10} \mathrm{C}=10^{-8} \mathrm{C}$.

Or:

$$
J=N q v=\left(10^{10} \mathrm{~m}^{-3}\right)\left(10^{-20} \mathrm{C}\right)(100 \mathrm{~m} / \mathrm{s})=10^{-8} \mathrm{~A} / \mathrm{m}^{2}
$$

Put in P, Maxwell equations become

$$
\begin{array}{ll}
\epsilon_{0} \nabla \cdot \mathbf{E}=-\nabla \cdot \mathbf{P} & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B}=\mu_{0} \frac{\partial \mathbf{P}}{\partial t}+\epsilon_{0} \mu_{0} \frac{\partial \mathbf{E}}{\partial t}
\end{array}
$$

Still need to specify $\mathbf{P}$
Expect $\Delta \mathbf{r} \propto \mathbf{E}$, therefore $\mathbf{P} \propto \mathbf{E}$
Write: $\mathbf{P}=\epsilon_{0} \chi \mathbf{E}$
with $\chi \equiv$ electric susceptability
(dimensionless, possibly complex)
Assumes steady state response

Non-linear optics:

$$
P=\epsilon_{0}\left(\chi E+\chi^{(2)} E^{2}+\chi^{(3)} E^{3}+\ldots\right)
$$

non-linear function
If $E$ is small, only linear term matters
Characteristic scale: electric field from nucleus

$$
E \sim \frac{e}{4 \pi \epsilon_{0} r^{2}} \approx 10^{11} \mathrm{~V} / \mathrm{m}
$$

Corresponds to $I \sim 10^{17} \mathrm{~W} / \mathrm{m}^{2}$ We'll assume $I \ll$ this

Want more? take Phys 532 next semester

We have $\mathbf{P}=\epsilon_{0} \chi \mathbf{E}$
Consider wave in infinite, uniform medium Then $\chi$ constant in space

So $\epsilon_{0} \nabla \cdot \mathbf{E}=-\nabla \cdot \mathbf{P}$ becomes $\nabla \cdot \mathbf{E}=-\chi \nabla \cdot \mathbf{E}$

Expect $\chi>0$, must have $\nabla \cdot \mathbf{E}=0$
same as vacuum!

But also $\nabla \times \mathbf{B}=\mu_{0} \frac{\partial \mathbf{P}}{\partial t}+\mu_{0} \epsilon_{0} \frac{\partial \mathbf{E}}{\partial t}$
becomes $\nabla \times \mathbf{B}=\mu_{0} \epsilon_{0}(\chi+1) \frac{\partial \mathbf{E}}{\partial t}$
Define electric permittivity $\epsilon=\epsilon_{0}(\chi+1)$

Then Maxwell equations become

$$
\begin{array}{ll}
\nabla \cdot \mathbf{E}=0 & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B}=\epsilon \mu_{0} \frac{\partial \mathbf{E}}{\partial t}
\end{array}
$$

Like vacuum, but $\epsilon_{0} \rightarrow \epsilon$

Still have waves, but
speed $c \rightarrow \frac{1}{\sqrt{\epsilon \mu_{0}}}=\frac{c}{\sqrt{1+\chi}}$
Define $\sqrt{1+\chi}=n$ index of refraction then $v=c / n$

Expect $\chi>0$, so $n>1$ and $v<c$

- Light is slower in medium than in vacuum
(Will see that's not always the case!)
Question: Since the electrons are displaced, and they have negative charge, shouldn't we normally expect $\chi<0$ ?

Effect on plane waves:
Still need $k=\omega / v$
Frequency $\omega$ doesn't change

So $k=n \omega / c \equiv n k_{0}$
$k_{0}=$ vacuum wave number
Wave number typically increases in medium
Wave vector $\mathbf{k}=n \mathbf{k}_{0}$

Have $\lambda=2 \pi / k=\lambda_{0} / n$
$\lambda_{0}=$ vacuum wavelength
Wavelength typically decreases in medium

Irradiance also changed:

$$
\text { Still } S=\frac{1}{2 \mu_{0}} \mathbf{E}_{0} \times \mathbf{B}_{0}
$$

Now $\mathbf{B}_{0}=\frac{1}{v} \widehat{\mathbf{k}} \times \mathbf{E}_{0}=\frac{n}{c} \widehat{\mathbf{k}} \times \mathbf{E}_{0}$
So

$$
\mathbf{S}=\frac{n}{2 \mu_{0} c}\left|\mathbf{E}_{0}\right|^{2} \widehat{\mathbf{k}}=\frac{n}{2 \eta_{0}}\left|\mathbf{E}_{0}\right|^{2} \hat{\mathbf{k}}
$$

and

$$
I=\frac{n}{2 \eta_{0}}\left|\mathbf{E}_{0}\right|^{2}
$$

Model for index (Hecht 3.5)

- Index of refraction is important

Could just measure for various materials, but can we relate it to a microscopic model of atom?

Quantitative accuracy: need quantum mechanics
Get basic idea with classical approach

Remember our atom model:


Displaced cloud feels linear restoring force (for small displacements)
Total force $\mathbf{F}=q \mathbf{E}-\kappa \Delta \mathbf{r}=m \frac{d^{2}}{d t^{2}} \Delta \mathbf{r}$
$\kappa=$ spring constant
$m=$ mass

Expect strong response ( $=$ large $n$ ) at $\omega \approx \omega_{0}=\sqrt{\kappa / m}$

For simplicity, take $\mathbf{E}$ polarized along $\widehat{\mathbf{x}}$ so $\Delta \mathbf{r} \rightarrow x$

Differential equation

$$
\ddot{x}+\omega_{0}^{2} x=\frac{q}{m} E(t)
$$

Simple harmonic oscillator infinite response at $\omega_{0}$

Should also include damping force Model with $\mathbf{F}_{\text {damp }}=-\beta \mathbf{v}$

Differential equation becomes

$$
\ddot{x}+\sigma \dot{x}+\omega_{0}^{2} x=\frac{q}{m} E(t)
$$

where $\sigma=\beta / m$

Damped harmonic oscillator

Solve for plane wave $E(t)=E_{0} e^{-i \omega t}$
look for solution $x(t)=x_{0} e^{-i \omega t}$
find $x_{0}=\frac{1}{\omega_{0}^{2}-i \omega \sigma-\omega^{2}} \frac{q}{m} E_{0}$
or generally $\Delta \mathbf{r}(t)=\frac{1}{\omega_{0}^{2}-i \omega \sigma-\omega^{2}} \frac{q}{m} \mathbf{E}(t)$
Typical resonance response

Gives dipole moment $\mathbf{p}(t)=q \Delta \mathbf{r}(t)$
and macroscopic polarization $\mathbf{P}(t)=N \mathbf{p}(t)$ :

$$
\mathbf{P}(t)=\frac{N q^{2}}{m} \frac{1}{\omega_{0}^{2}+i \omega \sigma-\omega^{2}} \mathbf{E}(t)
$$

By definition $\mathbf{P}=\epsilon_{0} \chi \mathbf{E}$, so

$$
\chi=\frac{N q^{2}}{\epsilon_{0} m} \frac{1}{\omega_{0}^{2}+i \omega \sigma-\omega^{2}}
$$

- Predicts macroscopic quantity $\chi$ in terms of microscopic quantities $q, m, \omega_{0}, \sigma$

Note, $\chi$ is complex for $\sigma \neq 0$.
Really just our complex representation What does it mean?
$n=\sqrt{1+\chi}$ is complex
write $\tilde{n}$ as reminder
So $\tilde{n}=n_{R}+i n_{I}$ and $\mathbf{k}=\tilde{n} \mathbf{k}_{0}$

Plane wave: $\mathbf{E}=\mathbf{E}_{0} e^{i\left[\left(n_{R}+i n_{I}\right) \mathbf{k}_{0} \cdot \mathbf{r}-\omega t\right]}$

$$
=\mathbf{E}_{0} e^{-n_{I} \mathbf{k}_{0} \cdot \mathbf{r}^{i}} e^{i\left(n_{R} \mathbf{k}_{0} \cdot \mathbf{r}-\omega t\right)}
$$

Really $\mathbf{E}=\left|E_{0}\right| \hat{\jmath} e^{-n_{I}} \mathbf{k}_{0} \cdot \mathbf{r} \cos \left(n_{R} \mathbf{k}_{0} \cdot \mathbf{r}-\omega t+\phi\right)$

Amplitude decays as wave propagates
models absorption
(Comes from damping in atoms)
Usually write $\tilde{n} \rightarrow n+i \frac{\alpha}{2 k_{0}}$
instead of $n_{R}+i n_{I}$

Say $\widehat{\mathbf{k}}=\hat{\mathbf{z}}$. Then $\mathbf{E}=\mathbf{E}_{0} e^{-\alpha z / 2} e^{i\left(n k_{0} z-\omega t\right)}$
and irradiance is $I=\frac{n}{2 \eta_{0}}\left|E_{0}\right|^{2} e^{-\alpha z}$
So $\alpha$ is absorption coefficient (units $\mathrm{m}^{-1}$ )
$I$ reduced by $1 / e$ in distance $1 / \alpha$


Also see that $n$ depends on $\omega$ :
Resonance at $\omega=\omega_{0}$

On resonance: high absorption

- bad for optics

Good materials: $\omega_{0}$ lies in UV

- gives high-frequency cutoff for transmission

Below resonance $n>1$ and $d n / d \omega>1$
so $v$ depends on $\omega$

- called dispersion

More on this later

Quantum mechanics gives similar result:

$$
\chi(\omega)=\frac{N e^{2}}{\epsilon_{0} m} \sum_{j} \frac{f_{j}}{\omega_{j}^{2}-\omega^{2}-i \omega \sigma_{j}}
$$

Main difference:
Many resonant frequencies $\omega_{j}$
(correspond to energy transitions)
Good optical materials: no resonances in visible

Weighting factors $f_{j}$ called oscillator strengths (related to transition matrix elements)

Index calculation has strange implications:
Since $n=n(\omega)$, wave velocity $v=v(\omega)$

- No longer have true wave equation
- Non-plane waves distorted in medium

Artifact of steady-state assumption

Predict possible to have $n<1$ : so $v>c$ ?

- Meaning of $v$ is tricky: still can't transmit info faster than $c$
- But pretty strange

Try to understand better next time

## Summary

- Electric field polarizes medium, causes current flow
- EM waves in medium are similar to in vacuum, with $v=c / n$
- Medium response exhibits resonances: absorption peaks
- Glass, other good materials have no resonances in optical region

