Light in Matter: Scattering Approach

Last time, talked about light in matter:

Include charge, current terms in Maxwell eqs
Try to only consider macroscopic effects
(= polarization $\mathbf{P}$)

Result: wave equation similar to vacuum
$\epsilon_0 \to \epsilon$
$c \to c/n$
Generally, waves slower in medium

Get n from microscopic model

Today: Take a different approach

- Consider direct effect of microscopic charges on field

Summarize:
Each atom radiates a new wave
total field = incident field + radiated field

Call radiated = scattered

Punch line:
Incident and scattered field both travel at $v = c$, but total field *looks* like it travels slower

Outline:

- Radiation
- Scattering by dense medium
- Scattering approach to index

Next time:

- Survey optical materials
- Start considering boundaries between media

Radiation (Hecht 3.4)

Want to consider sources explicitly:
simplest source = radiating charge

(Also, nice to know where light comes from!)

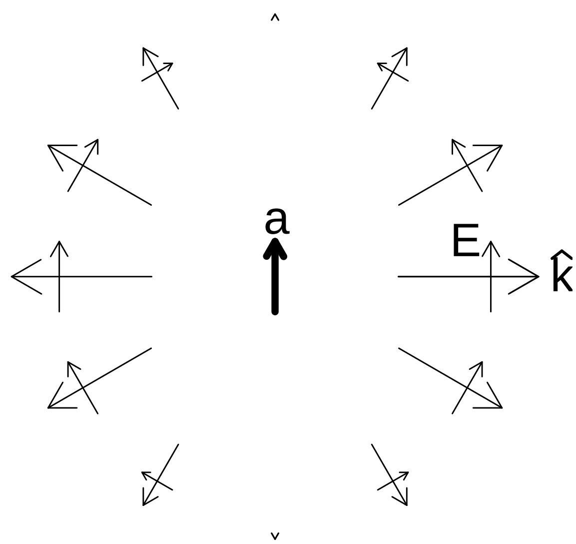
Basic result:
Accelerating charge emits EM wave

Why? Hecht gives nice explanation
see Figure 3.28 and discussion, pg. 59

General characteristics:

- Light radiated $\perp$ acceleration $\mathbf{a}$
- $\mathbf{E}$ polarized along $\mathbf{a}$ (but $\perp \mathbf{k}$)

Picture:



More precise, solve Maxwell eqns

Simple setup:

oscillating dipole $\mathbf{p} = p_0 \hat{\mathbf{z}} \exp(-i\omega t)$
located at $\mathbf{r} = 0$

But math is hard... for derivation, see
Jackson, *Classical Electrodynamics*
Section 16.2

Result:

$$\mathbf{E} = \frac{p_0 k^3}{4\pi\epsilon_0} e^{i(kr-\omega t)} \times$$
$$\left\{ \left[-\frac{1}{kr} - \frac{3i}{(kr)^2} + \frac{3}{(kr)^3} \right] \left(\frac{xz}{r^2}\hat{\mathbf{x}} + \frac{yz}{r^2}\hat{\mathbf{y}} - \frac{x^2+y^2}{r^2}\hat{\mathbf{z}} \right) \right.$$
$$\left. - 2 \left[\frac{i}{(kr)^2} - \frac{1}{(kr)^3} \right] \hat{\mathbf{z}} \right\}$$

$$\mathbf{B} = \frac{p_0 k^3}{4\pi\epsilon_0 c} e^{i(kr-\omega t)} \left\{ \left[\frac{1}{kr} + \frac{i}{(kr)^2} \right] \left(\frac{y}{r}\hat{\mathbf{x}} - \frac{x}{r}\hat{\mathbf{y}} \right) \right\}$$

Real solutions are very complicated!

Optics: always assume $kr = 2\pi r/\lambda \gg 1$:

$$\mathbf{E} \to -\frac{k^3}{4\pi\epsilon_0} p_0 \frac{e^{i(kr-\omega t)}}{kr} \sin\theta\, \hat{\theta}$$

$$\mathbf{B} \to -\frac{k^3}{4\pi\epsilon_0 c} p_0 \frac{e^{i(kr-\omega t)}}{kr} \sin\theta\, \hat{\phi}$$

Spherical coords (r, θ, ϕ)

Called *dipole radiation field*

$\approx$ spherical wave, with extra $\sin\theta$ factor

Scattering (Hecht 4.2)

Think about plane wave in matter
    atoms $\rightarrow$ oscillating dipoles
    $\rightarrow$ radiation

Try to understand effect of radiated field

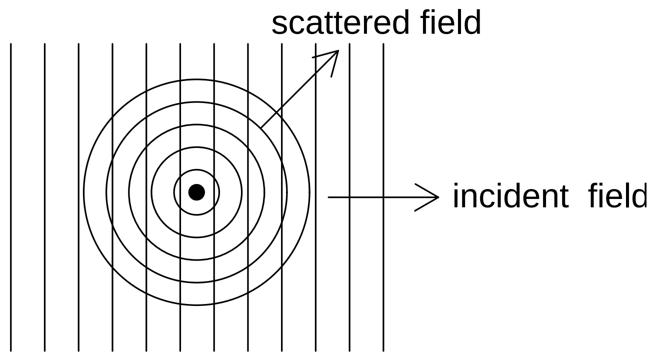
I'll give more mathematical derivation:
    Hecht gives more conceptual argument

Start:
Plane wave incident on single atom:
    induce dipole $\mathbf{p} = \epsilon_0 \chi_1 \mathbf{E}$

with $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$
and $\chi_1 =$ "single-atom susceptability"
    $= \chi/N \qquad (N = \text{density})$

Atom produces dipole field $\approx$ spherical wave
    centered at atom location

Draw wave fronts:



Fields add:

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{incident}} + \mathbf{E}_{\text{scattered}}$$

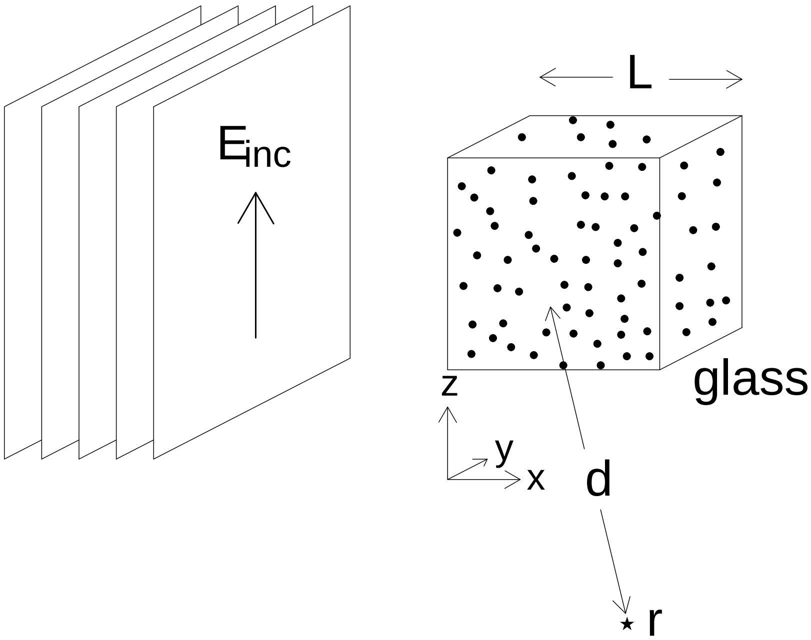
Real medium has many atoms

many $\mathbf{E}_{\text{scat}}$'s

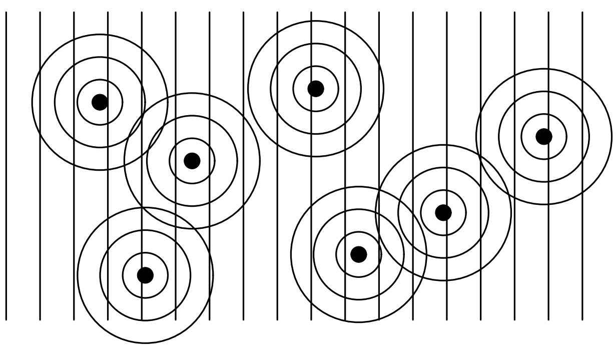
First model: scattering by glass cube

- Glass: atoms randomly distributed
- Assume cube size $L \gg \lambda$
- Measure at distance $d \gg L$
- Incident field: $\mathbf{E} = E_0 \hat{\mathbf{z}} e^{i(kx - \omega t)}$
  take $x = 0$ at front face of cube

Setup:



Need to add up scattered fields from each atom



Will see that fields tend to cancel out,
except when $\mathbf{r}$ is in *front* of medium

Write $\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{inc}}(\mathbf{r}) + \sum_j \mathbf{E}_j(\mathbf{r})$

$\mathbf{E}_j(\mathbf{r})$ = scattered field from atom j

Calculate $\mathbf{E}_j$:

dipole $\mathbf{p}_j = \epsilon_0 \chi_1 \mathbf{E}_{\text{inc}}(\mathbf{r}_j)$

$$= \epsilon_0 \chi_1 E_0 \hat{\mathbf{z}} e^{i(kx_j - \omega t)}$$

where $\mathbf{r}_j = (x_j, y_j, z_j)$ is position of atom j

Produces scattered field

$$\mathbf{E}_j(\mathbf{r}) = -\frac{k^3}{4\pi\epsilon_0} p_{j0} \frac{e^{i(kd_j - \omega t)}}{kd_j} \sin\theta_j \, \hat{\theta}_j$$

where $d_j = |\mathbf{r} - \mathbf{r}_j|$ is distance to atom j

Use expression for $\mathbf{p}_j$:

$$\mathbf{E}_j(\mathbf{r}) = -\frac{k^3}{4\pi\epsilon_0} \left(\epsilon_0 \chi_1 E_0 e^{ikx_j} \right) \frac{e^{i(kd_j - \omega t)}}{kd_j} \sin\theta_j \, \hat{\theta}_j$$

Use $d \gg L$ to replace $d_j \to d$

but only outside of exponent...

Inside needs higher precision since k is large

Also take $\theta_j \to 90^\circ$ (so $\hat{\theta}_j \to -\hat{\mathbf{z}}$)

for simplicity

Obtain contribution of atom j to field at $\mathbf{r}$:

$$\boxed{\mathbf{E}_j(\mathbf{r}) = \frac{k^2 \chi_1 E_0}{4\pi d} \hat{\mathbf{z}} e^{i[k(x_j + d_j) - \omega t]}}$$

x_j = distance from atom to front of cube

d_j = distance from atom to $\mathbf{r}$

Need to sum this over all j

Note x_j, d_j vary by $\sim L$ across cube

Since $L \gg \lambda$, $\phi_j \equiv k(x_j + d_j)$ varies over many π

equally likely + or -

fields tend to cancel out

Show cancellation explicitly:

consider $f = \sum_j e^{i\phi_j}$

Have $\mathbf{E} \propto f$

Assume each ϕ_j random

$\langle \ldots \rangle$ = average over possible values

Then

$$\langle f \rangle = \sum_j \left\langle e^{i\phi_j} \right\rangle = \mathcal{N} \left\langle e^{i\phi} \right\rangle$$

for $\mathcal{N}$ = number of atoms = NL^3

But

$$\left\langle e^{i\phi} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i\phi} \, d\phi$$
$$= \frac{1}{2\pi i} \left(e^{i2\pi} - e^0 \right) = 0$$

So $\langle f \rangle = 0 = \langle \mathbf{E} \rangle$

But cancellation not perfect

Irradiance $I \propto |E|^2$

so $\langle I \rangle \propto \left\langle |E|^2 \right\rangle \neq 0$

Get $\left\langle |f|^2 \right\rangle = \left\langle \sum_j e^{i\phi_j} \sum_\ell e^{-i\phi_\ell} \right\rangle$
$$= \sum_{j\ell} \left\langle e^{i(\phi_j - \phi_\ell)} \right\rangle$$

If $j \neq \ell$, average is zero as before

If $j = \ell$, average = 1

So $\left\langle |f|^2 \right\rangle = \mathcal{N}$

So rms scattered field at $\mathbf{r} \propto \mathcal{N}^{1/2}$

Means scattered field *per atom* decreases like $\mathcal{N}^{-1/2}$

In a dense medium, scattering is suppressed.

But still have $I_{\text{scat}} \propto \mathcal{N}$
seems like what you would expect!

Called *Rayleigh scattering*: why sky is blue

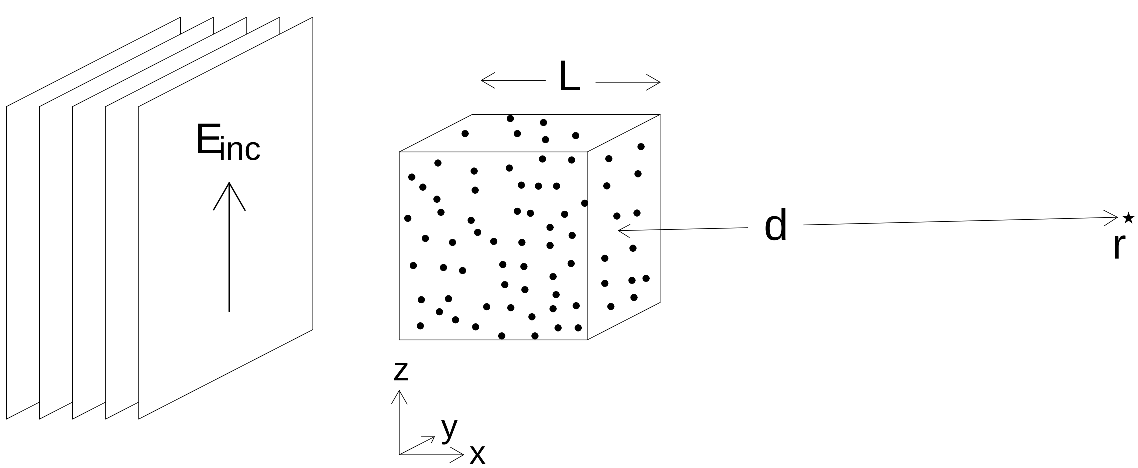
Different from:

- Case $L \lesssim \lambda^3$, phases don't cancel
  get large scattering amplitudes: superradiance

- *Forward scattering*

Forward Scattering

Consider scattered field in front of medium:



Difference: now x_j and d_j correlated

Set d = distance from $\mathbf{r}$ to back of cube

if $d \gg L$, have $d_j \approx d + L - x_j$

Phase $\phi_j = k(d_j + x_j) \approx k(d + L) = kx$
doesn't depend on j

Scattered fields don't cancel out:

$$\sum_j \mathbf{E}_j(\mathbf{r}) = \sum_j \frac{k^2 \chi_1 E_0}{4\pi d} e^{i[k(x_j + d_j) - \omega t]}$$
$$= \sum_j \frac{k^2 \chi_1 E_0}{4\pi d} e^{i(kx - \omega t)}$$
$$= \mathcal{N} \frac{k^2 \chi_1}{4\pi d} E_0 e^{i(kx - \omega t)}$$

So $\mathbf{E}_{\text{scat}}$ scales as $\mathcal{N}$ and $I \propto \mathcal{N}^2$!

Forward scattering is *strong*

Question: We get strong forward scattering because the phases ϕ_j from the different atoms are all the same. This happens because the two components of the phase, x_j and d_j, are correlated. But x_j and d_j are also correlated for backwards scattering... should we expect a strong backwards scattering effect?

Hint: $\phi_j = k(d_j + x_j)$, and for backwards scattering, $d_j \approx d - L + x_j$

Note $\mathbf{E}_{\text{scat}}(\mathbf{r}) = \beta \mathbf{E}_{\text{inc}}(\mathbf{r})$ for

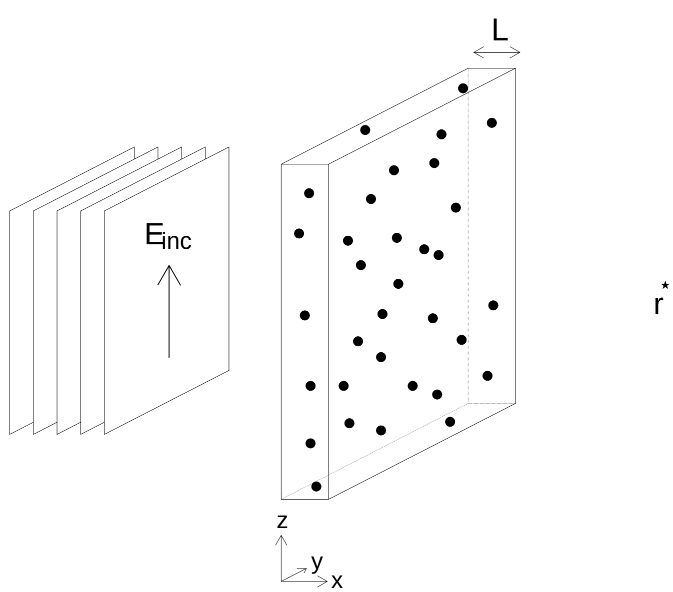
$$\beta = \frac{k^2 \chi_1 \mathcal{N}}{4\pi d}$$

forward scattered field at $\mathbf{r} \propto$ incident field at $\mathbf{r}$

Use $\chi_1 = \chi/N = \chi V/\mathcal{N}$
with $V = L^3 =$ volume of cube
Rewrite

$$\beta = \frac{k^2 \chi V}{4\pi d}$$

Did calculation for small cube
Size effects related to diffraction, more later
For now, generalize to infinite medium: glass slab

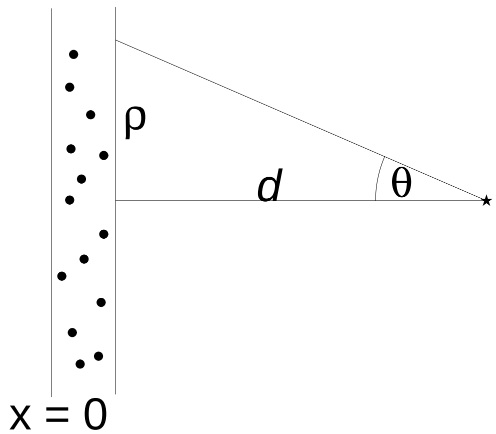


Slab thickness $L \ll d$

Front edge at $x = 0$

Measure at $x = d + L$

Now have forward scattering from some atoms,
not from others



Depends on angle θ, $\perp$ distance ρ

Have $d_j = \sqrt{(d + L - x_j)^2 + \rho_j^2}$

For small ρ_j, use $(1+\epsilon)^{1/2} \approx 1 + \epsilon/2$

$$\text{Then } d_j \approx d + L - x_j + \frac{\rho_j^2}{2(d + L - x_j)}$$

$$\approx d + L - x_j + \frac{\rho_j^2}{2d}$$

(using $d \gg L, x_j$)

For forward scattering, need $\phi_j \approx$ constant, independent of j

$$\phi_j = k(d_j + x_j) \approx k(d + L) + \frac{k\rho_j^2}{2d}$$

Estimate ϕ_j can vary by $\sim$1 radian

Get forward scattering from atoms with

$$\rho_j < \rho_{\text{max}} = \sqrt{\frac{2d}{k}}$$

Defines volume $V_{\text{eff}} \approx \pi\rho_{\text{max}}^2 L = \dfrac{2\pi L d}{k}$

Atoms in V_{eff} contribute to forward scattering

Put this volume into formula for scattered field:

$\mathbf{E}_{\text{scat}} = \beta \mathbf{E}_{\text{inc}}$ with

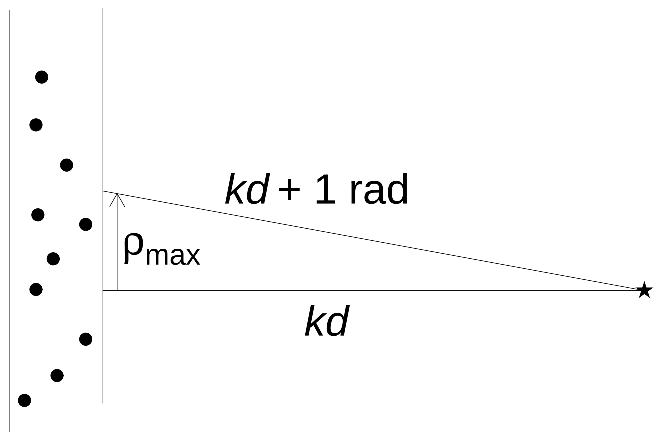
$$\beta = \frac{k^2 \chi V_{\text{eff}}}{4\pi d} = \frac{k^2\chi}{4\pi d}\left(\frac{2\pi L d}{k}\right) = \frac{kL}{2}\chi$$

Question: How can d drop out of β? Surely, the further you are from the atoms, the weaker the scattered field should be!

Still missing one effect:

most atoms at edge of volume, have $d_j > d + L - x_j$

On average, contribute additional phase shift



Won't go through calculation (see Jackson §9.14)

Result: extra factor of i

$$\mathbf{E}_{\text{scat}} = i\frac{kL}{2}\chi\mathbf{E}_{\text{inc}}$$

Now go back to total field at $\mathbf{r}$:

$$\begin{aligned}\mathbf{E}_{\text{tot}}(\mathbf{r}) &= \mathbf{E}_{\text{inc}}(\mathbf{r}) + \mathbf{E}_{\text{scat}}(\mathbf{r}) \\ &= \mathbf{E}_{\text{inc}}(\mathbf{r})\left(1 + i\frac{1}{2}kL\chi\right)\end{aligned}$$

We've implicitly assumed that scattering is weak (otherwise, scattered field would be re-scattered)

So we are limited to $kL\chi \ll 1$

Then we can rewrite $\mathbf{E}_{\text{tot}}$

$$\begin{aligned}\mathbf{E}_{\text{tot}}(\mathbf{r}) &= \mathbf{E}_{\text{inc}}(\mathbf{r})(1 + ikL\chi/2) \\ &= \mathbf{E}_{\text{inc}}(\mathbf{r})e^{ikL\chi/2}\end{aligned}$$

Since $e^x \approx 1 + x$ for $|\epsilon| \ll 1$

So have

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_0 e^{i[k(L+d)-\omega t]} e^{ikL\chi/2}$$

Rearrage exponents:

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = \mathbf{E}_0 e^{i(kd-\omega t)} e^{ikL(1+\chi/2)}$$

For $\chi \ll 1$, recognize

$$1 + \frac{\chi}{2} \approx \sqrt{1+\chi} = n \text{ index of refraction}$$

So $\mathbf{E}_{\text{tot}}(\mathbf{r}) = \mathbf{E}_0 e^{i(kd-\omega t)} e^{inkL}$

Final result:

$$\mathbf{E}_{\text{tot}} = E_0 e^{i[(nkL+kd)-\omega t)}$$

So $k \to nk$ for distance L, then k for distance d

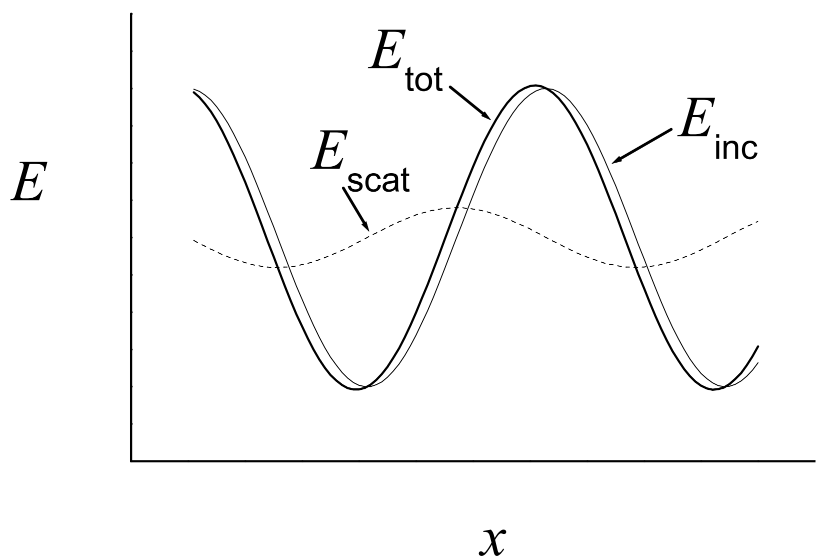
Like $k \to nk$ in medium:

Same as $v = c/n$ in medium

Same result as before!

But here all waves travel at speed c!?

Graphically:



Scattered field makes total field lag a bit

Upshot:

- Have "rederived" index of refraction
- Now waves travel at speed c always

- Only *looks* like wave is slower in medium
  due to interference by scattered wave

Did calc for weak scattering, does generalize

So any time we see funny effect of index,
remember its really a scattering effect

Question:

How can we explain absorption in this scattering picture?

Hint:

If $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scat}}$, how can we make $\mathbf{E}_{\text{inc}}$ go away?

Can explain *any* optical effect in terms of scattered fields

But usually much easier to use index approach, don't need to explicitly calculate scattered fields

Summary:

- Accelerating charges radiate

- Radiated fields from medium usually cancel, gives $I_{\text{scat}} \propto$ number of atoms

- In forward direction, radiated fields add, gives $E_{\text{scat}} \propto$ number of atoms

- Sum of incident field and forward scattered field gives total field with delayed phase: looks slow