Fresnel Relations

Last time, starting looking at how light propagates across boundaries.

Scattering idea → law of reflection, Snell's law
Generalized to Fermat's principle

Important question:
How much light will be reflected vs transmitted?

Answer today using Maxwell equations

Outline:

- Boundary conditions for $\mathbf{E}$ and $\mathbf{B}$
- Fresnel equations
- Brewster's angle
- Reflectance and transmittance

Everything today from Hecht 4.6

Next time: when the Fresnel equations become complex

Boundary Conditions

Maxwell equations in medium:

$$\epsilon_0 \nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\mathbf{P}$ = macroscopic polarization = $\epsilon_0 \chi \mathbf{E}$

Now consider $\chi = \chi(\mathbf{r})$

Rewrite:

$$\epsilon_0 \nabla \cdot [(1+\chi)\mathbf{E}] = 0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial}{\partial t}\Big[(1+\chi)\mathbf{E}\Big] \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Recall $\epsilon_0(1+\chi) \equiv \epsilon$, electric permittivity

Convenient to define $\mathbf{D} = \epsilon \mathbf{E}$
"Electric displacement" (units C/m^2)

Then have

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \frac{\partial \mathbf{D}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

"Hides" charges of medium, even when crossing boundaries

Works for conductors too, if $\epsilon \to$ complex

Know how wave propagates in uniform medium

Don't know how to relate $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$ on opposite sides of boundary

Easiest to go back to integral form:

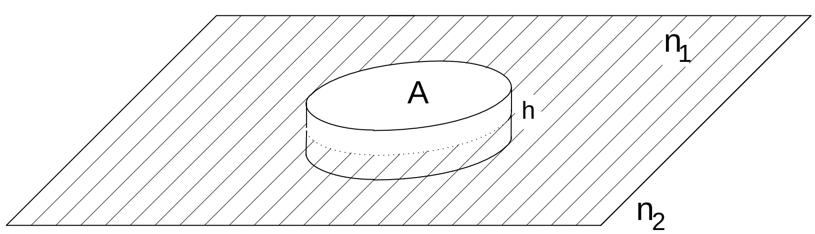
If $\nabla \cdot \mathbf{D} = 0$, then $\oiint \mathbf{D} \cdot \mathbf{dS} = 0$

See what this says about boundary

Make little pillbox surface on boundary:

area A small but nonzero

height $h \to 0$



Then $\oiint \mathbf{D} \cdot \mathbf{dS} \to A(D_{1\perp} - D_{2\perp}) = 0$

where $D_\perp$ = compenent of $\mathbf{D}$ normal to boundary

So $D_\perp$ is continuous across boundary

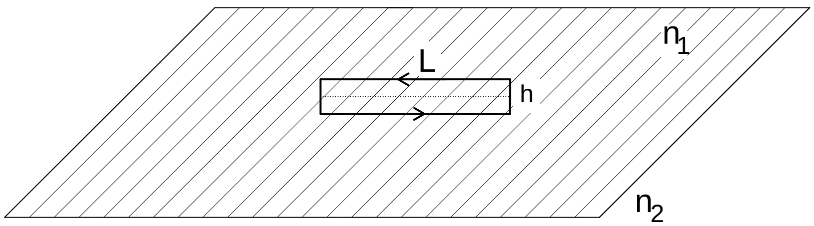
Also have

$$\oint \mathbf{E} \cdot \mathbf{dl} = - \iint \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$$

Make little loop normal to boundary

length L small

height $h \to 0$



Then $\oint \mathbf{E} \cdot \mathbf{dl} \to L(E_{1\|} - E_{2\|})$

$E_{\|}$ = component of $\mathbf{E}$ parallel to boundary

And

$$\iint \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} \to \frac{Lh}{2} \left(\frac{\partial B_{1\|}}{\partial t} + \frac{\partial B_{2\|}}{\partial t} \right)$$

$$= 0 \quad \text{for } h \to 0$$

So $\mathbf{E}_{\|}$ is continuous across boundary

Similarly, from

$$\oiint \mathbf{B}\cdot\mathbf{dS} = 0 \text{ and } \oint \mathbf{B}\cdot\mathbf{dl} = \mu_0 \iint \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{dS}$$

show that $B_\perp$ and $B_\parallel$ are continous

So $\mathbf{B}$ is same on either side of boundary

Note: only true for nonmagnetic materials
(normal in optics)
Hecht gives general formulas

Apply continuity conditions to boundary $n_i \to n_t$

Three fields:

Incident $\mathbf{E}_i = \mathbf{E}_{i0}e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}$

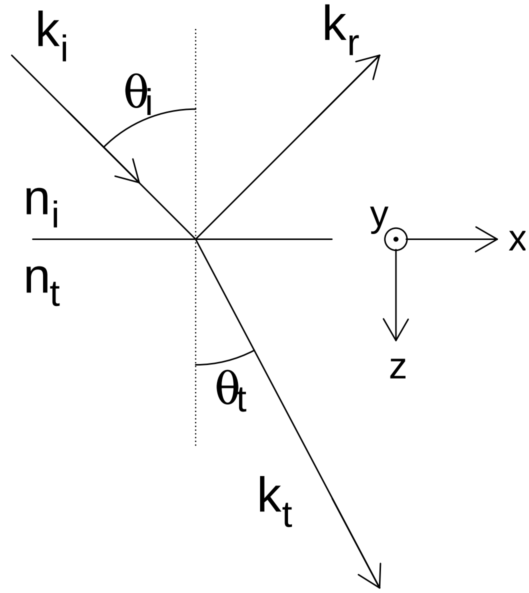
Reflected $\mathbf{E}_r = \mathbf{E}_{r0}e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}$

Transmitted $\mathbf{E}_t = \mathbf{E}_{t0}e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}$

Here $\mathbf{k}_i = n_i\frac{\omega}{c}\widehat{\mathbf{k}}_i$ etc. Also have $\mathbf{B} = \frac{n}{c}\widehat{\mathbf{k}}\times\mathbf{E}$ for each

Want to determine $\mathbf{E}_r$ and $\mathbf{E}_t$ if given $\mathbf{E}_i$

Set up coordinates so $k_y = 0$ always



At $z = 0$, have:

$$\mathbf{B}_i + \mathbf{B}_r = \mathbf{B}_t$$

$$E_{ix} + E_{rx} = E_{tx}$$

$$E_{iy} + E_{ry} = E_{ty}$$

$$n_i^2 E_{iz} + n_i^2 E_{rz} = n_t^2 E_{tz}$$

Question: Wait, where did those n^2's come from?

E_x equation says

$$E_{i0x} e^{i(k_{ix}x - \omega t)} + E_{r0x} e^{i(k_{rx}x - \omega t)} = E_{t0x} e^{i(k_{tx}x - \omega t)}$$

for all x

Only possible if $k_{ix} = k_{rx} = k_{tx}$

Implies $\sin\theta_i = \sin\theta_r$ and $n_i \sin\theta_i = n_t \sin\theta_t$

- Gives law of reflection, Snell's law

So x dependence drops out, leaves equations for amplitudes $\mathbf{E}_0$, $\mathbf{B}_0$

example: $E_{i0x} + E_{r0x} = E_{t0x}$

Easiest to separate two cases:

Case I: $E_{i0x} = 0$

Then all E_x components = 0

So $\mathbf{E}_0$'s are $\perp$ to plane of incidence

Called "s-polarized" or "TE-polarized"

Case II: $E_{i0y} = 0$

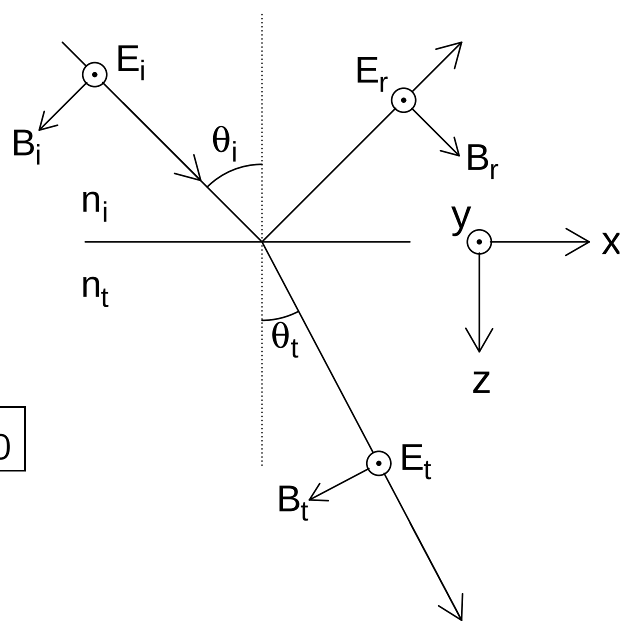
Then all E_y components = 0

So $\mathbf{E}_0$'s are in plane of incidence

Called "p-polarized" or "TM-polarized"

Can write general wave as superposition of these

13



Case I: $\mathbf{E} \propto \hat{\mathbf{y}}$

So $\mathbf{B}$ in xz-plane

Continuity:

$E_{i0y} = E_{i0}$ etc,

so $\boxed{E_{i0} + E_{r0} = E_{t0}}$

For $\mathbf{B}$:

$$B_{i0x} = -B_{i0} \cos\theta_i$$
$$B_{r0x} = B_{r0} \cos\theta_i$$
$$B_{t0x} = -B_{t0} \cos\theta_t$$

Have $B_{i0x} + B_{r0x} = B_{t0x}$, and

$$B_{i0x} = \frac{n_i}{c} E_{i0} \text{ etc.}$$

so

$$\boxed{-n_i E_{i0} \cos\theta_i + n_i E_{r0} \cos\theta_i = -n_t E_{t0} \cos\theta_t}$$

Two equations, two unknowns E_{r0} and E_{t0}
(B_z equation is redundant)

Solve:

$$\begin{aligned}(E_{i0} - E_{r0}) n_i \cos\theta_i &= E_{t0} n_t \cos\theta_t \\ &= (E_{i0} + E_{r0}) n_t \cos\theta_t\end{aligned}$$

$$E_{i0}(n_i \cos\theta_i - n_t \cos\theta_t) = E_{r0}(n_i \cos\theta_i + n_t \cos\theta_t)$$

Write $E_{r0} = r_\perp E_{i0}$ for

$$\boxed{r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

$r_\perp$ = amplitude reflection coefficient
(for TE polarization)

Then get E_{t0}:

$$(E_{i0} - E_{r0}) n_i \cos\theta_i = E_{t0} n_t \cos\theta_t$$

$$E_{t0} = \frac{n_i \cos\theta_i}{n_t \cos\theta_t}(1 - r_\perp) E_{i0}$$

Define $E_{t0} = t_\perp E_{i0}$

$$t_\perp = \frac{n_i \cos\theta_i}{n_t \cos\theta_t}(1 - r_\perp)$$

$$= \frac{n_i \cos\theta_i}{n_t \cos\theta_t} \frac{2 n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

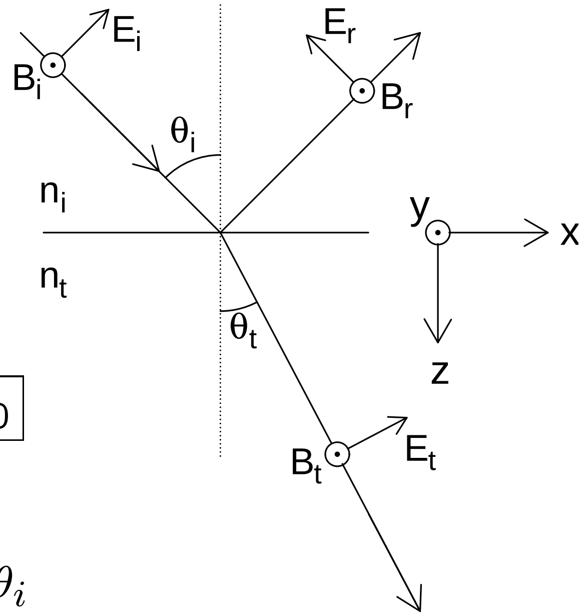
So $$\boxed{t_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}}$$

amplitude transmission coefficient
(s-polarization)

This solves case I

Question: What happens to $r_\perp$ and $t_\perp$ if $n_i = n_t$?

Case II:



Now have $\mathbf{B} \perp$ plane

$$B_{i0} + B_{r0} = B_{t0}$$

or $\boxed{n_i E_{i0} + n_i E_{r0} = n_t E_{t0}}$

For $\mathbf{E}$: $E_{i0x} = E_{i0} \cos\theta_i$

$$E_{r0x} = -E_{r0} \cos\theta_i$$

$$E_{t0x} = E_{t0} \cos\theta_t$$

$$\boxed{E_{i0} \cos\theta_i - E_{r0} \cos\theta_i = E_{t0} \cos\theta_t}$$

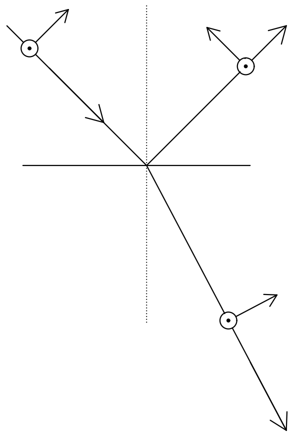
Solve, get $E_{r0} = r_\parallel E_{i0}$, $E_{t0} = t_\parallel E_{i0}$

$$\boxed{r_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i}}$$

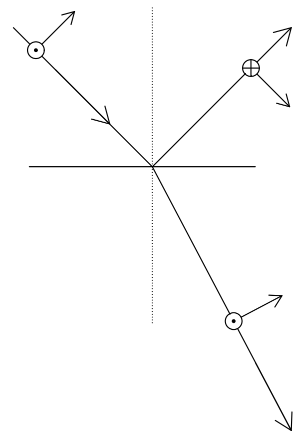
$$\boxed{t_\parallel = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}}$$

Call r's, t's Fresnel coefficients, equations are *Fresnel relations*

Note: signs depend on picture set up



vs.



Gives opposite sign for r's

Hecht's set up most common

Question: At normal incidence $r_\perp = -r_\parallel$. How are the actual directions of $\mathbf{E}_{\text{inc}}$ and $\mathbf{E}_{\text{ref}}$ related?

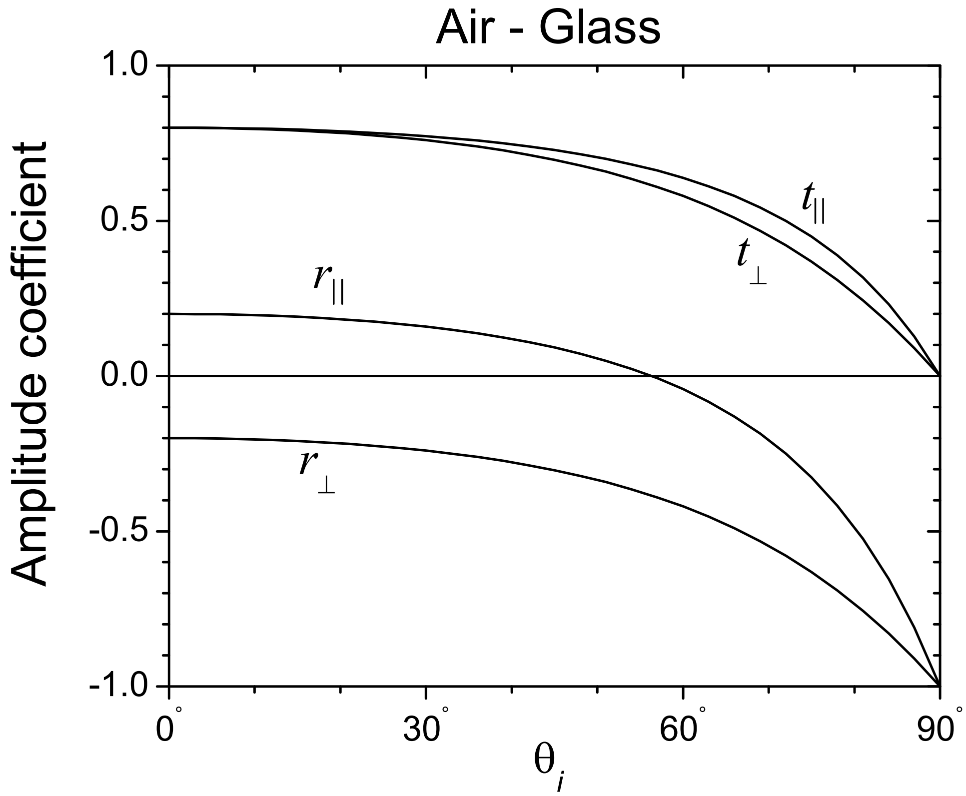
For now consider $n_i < n_t$: "external incidence"

Plot for air ($n_i = 1$) $\rightarrow$ glass ($n_t = 1.5$)

Need to use Snell's Law to get θ_t

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t}$$
$$= \sqrt{1 - \frac{n_i^2}{n_t^2}\sin^2\theta_i}$$

Question: Do we need to worry about $\pm$ with square root?



Features:

- $r_\perp = -r_\parallel$ at $\theta_i = 0$

No physical difference between TE and TM
Picture difference gives minus sign

- $r \to 1$ as $\theta_i \to 90^\circ$

Everything reflects at glancing incidence

- $r_\parallel \to 0$ at $\theta_i = \theta_p$

Usually called Brewster's angle
Hecht calls "polarization angle"

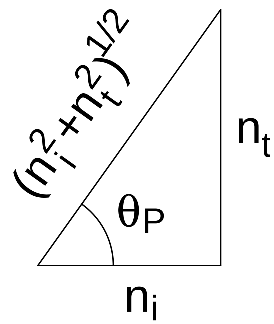
Demo!

Brewster's angle important for lasers
best way to minimize reflections

Solve $r_{\|} = 0$: $n_t \cos\theta_p = n_i \cos\theta_t$

Get $\sin\theta_p = \dfrac{n_t}{\sqrt{n_i^2 + n_t^2}}$

Get some insight:



See $\tan\theta_p = n_t/n_i$

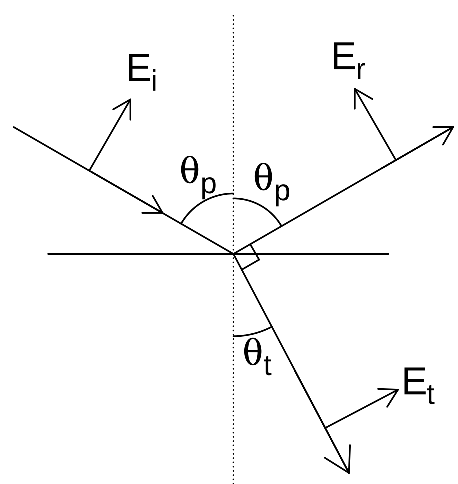
So $n_i \sin\theta_p = n_t \cos\theta_p$

But $n_t \cos\theta_p = n_i \cos\theta_t$
so $\sin\theta_p = \cos\theta_t \quad \Rightarrow \quad \theta_p + \theta_t = 90^\circ$

Picture:

Atoms in transmitted medium oscillate along $\mathbf{E}_t$

Dipole radiation $\rightarrow 0$ in direction of oscillation



Brewster's angle:
when direction of oscillation $= \hat{\mathbf{k}}_{\text{reflect}}$

For air $\rightarrow$ glass, $\theta_p = 56.3^\circ$

Note, r and t are amplitude coefficients:
give E-fields

Usually more interested in transmitted and reflected power P

Define *reflectance* $R = P_{\text{ref}}/P_{\text{inc}}$
transmittance $T = P_{\text{trans}}/P_{\text{inc}}$

Get P from Poynting vector $\mathbf{S}$:

Plane waves: $\mathbf{S} = \dfrac{n}{2\eta_0}|E_0|^2\,\hat{\mathbf{k}}$

Power through area $dA = \mathbf{S}\cdot\hat{\mathbf{u}}\,dA$
$\hat{\mathbf{u}}$ = normal to surface
here $\hat{\mathbf{k}}\cdot\hat{\mathbf{u}} = \cos\theta$

So $P_{\text{inc}} = \dfrac{n_i}{2\eta_0}|E_{i0}|^2\cos\theta_i\,dA$

$$P_{\text{refl}} = \frac{n_i}{2\eta_0}|E_{j0}|^2\cos\theta_i\,dA$$

$$P_{\text{trans}} = \frac{n_t}{2\eta_0}|E_{t0}|^2\cos\theta_t\,dA$$

Then

$$R = \frac{P_{\text{ref}}}{P_{\text{inc}}} = \frac{|E_{r0}|^2}{|E_{i0}|^2} = |r|^2$$

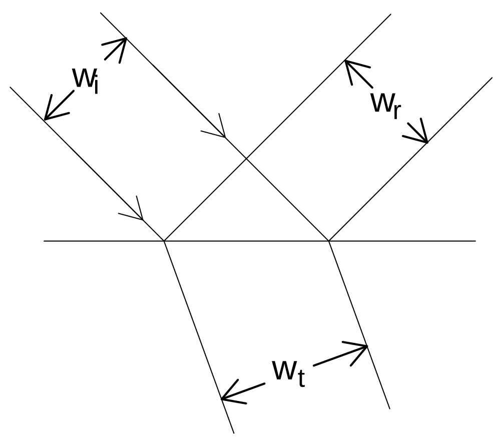
and

$$T = \frac{P_{\text{trans}}}{P_{\text{inc}}} = \frac{n_t \cos\theta_t |E_{t0}|^2}{n_i \cos\theta_i |E_{i0}|^2} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} |t|^2$$

Extra factors in T make sense:

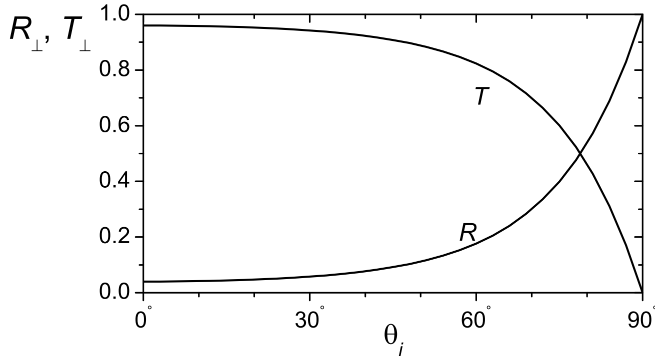
- n accounts for difference in speed
- $\cos\theta$ accounts for difference in area

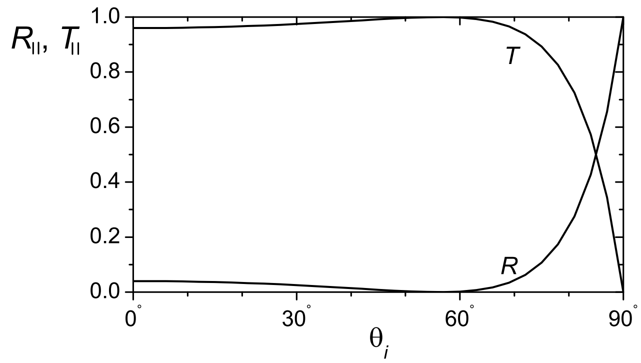
Here $w_t > w_i$
Irradiance decreases even
if all power transmitted



Can show $R + T = 1$ for both $\perp$ and $\parallel$ cases

- Energy conserved (if n is real)





Summary:

- Maxwell equations give continuity relations
- Fresnel coefficients r, t relate $\mathbf{E}_{\text{inc}}$, $\mathbf{E}_{\text{ref}}$, $\mathbf{E}_{\text{trans}}$
- Two cases $\perp$ (= TE = s) and $\|$ (= TM = p) are different
- TM case exhibits Brewster's angle, $r(\theta_p) = 0$
- Fresnel coeffs related to power reflectance R, transmittance T
- Air-glass boundary reflects 4% at $\theta_i = 0$