Total Internal Reflection & Metal Mirrors

Last time, derived Fresnel relations

Give amplitude of reflected, transmitted waves at boundary

Focused on simple boundaries: air $\rightarrow$ glass

Today, consider more complicated situations

- Total internal reflection
  - Evanescant waves
- Materials with complex index

This will wrap up unit on fundamental theory

Next time: ray optics

Total Internal Reflection (Hecht 4.7)

So far, considered $n_i < n_t$

If $n_i > n_t$, problem with Snell's Law:

$$\sin\theta_t = \frac{n_i}{n_t}\sin\theta_i$$

What if $(n_i/n_t)\sin\theta_i > 1$?

For instance, glass ($n_i = 1.5$) $\rightarrow$ air ($n_t = 1$):

$$\text{if } \theta_i > 41.8^\circ\text{, then } \sin\theta_i > \frac{1}{1.5}$$

Demo!

For $\boxed{\sin\theta_i > \sin\theta_c \equiv n_t/n_i}$, all light is reflected

Called *total internal reflection* = TIR

only occurs when light exits medium

(high $n \rightarrow$ low n)

Would like to understand from Fresnel relations

$$r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

How can we use? Don't have a θ_t!

Trick: use complex θ_t

Say $\theta_t = a + ib$

Considered $\cos\theta_t$ in homework 1
Look at $\sin\theta_t$ now

Use

$$\sin(a + ib) = \sin(a)\cos(ib) + \cos(a)\sin(ib)$$

Just need $\sin(ib)$, $\cos(ib)$

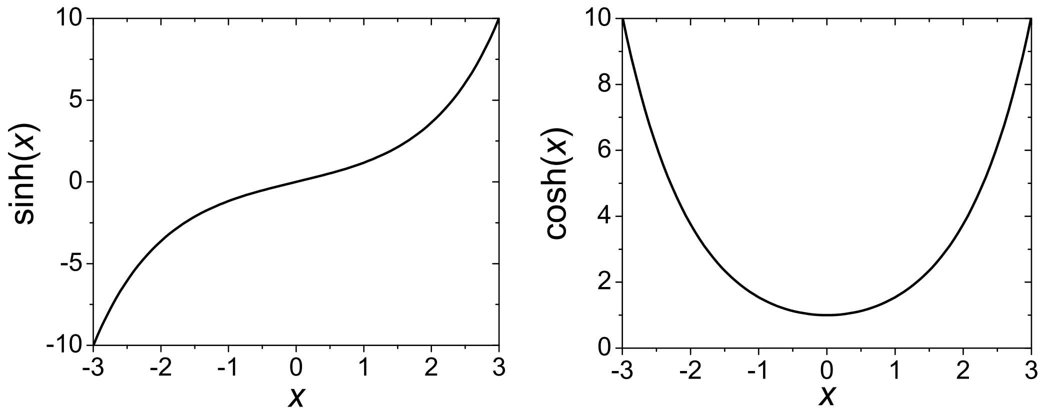
Euler identity gives: $$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$
$$\cos(x) = \frac{1}{2}\left(e^{ix} - e^{-ix}\right)$$

So $$\sin(ib) = \frac{1}{2i}\left(e^{i(ib)} - e^{-i(ib)}\right) = -\frac{i}{2}\left(e^{-b} - e^{b}\right)$$

Hyperbolic sine: $\sinh(x) \equiv \dfrac{1}{2}\left(e^{x} - e^{-x}\right)$

So $\sin(ib) = i\sinh(b)$

Hyperbolic sine and cosine:



Also

$$\begin{aligned}\cos(ib) &= \frac{1}{2}\left(e^{i(ib)} + e^{-i(ib)}\right)\\ &= \frac{1}{2}\left(e^{-b} + e^{b}\right)\\ &\equiv \cosh(b)\end{aligned}$$

Hyperbolic cosine

So: $\sin(a+ib) = \sin(a)\cosh(b) + i\cos(a)\sinh(b)$

For large b, $\sinh b, \cosh b \to e^b$

no problem satisfying $n_i \sin\theta_i = n_t \sin\theta_t$

Want $n_i \sin\theta_i = n_t \sin\theta_t$ with complex θ_t

Assume $n_i \sin\theta_i$ and n_t real:

Then Im $[n_t \sin\theta_t] = n_t \cos(a) \sinh(b) = 0$

Don't want $b = 0$, so take $a = \frac{\pi}{2}$

Gives $\sin\theta_t = \cosh b$

Snell's law becomes

$$n_i \sin\theta_i = n_t \cosh b$$

Note $\cosh b > 1$, require $\sin\theta_i > n_t/n_i \equiv \sin\theta_c$

Example: What is the complex transmission angle for light propagating from glass to air with an angle of incidence of 60°?

If $\theta_i = 60^\circ$ then $b = \cosh^{-1}[1.5 \sin(60^\circ)] = 0.755$

So $\theta_t = \dfrac{\pi}{2} + 0.755i$

Just need a good calculator!

Question: What are the units of b?

What happens to Fresnel coefficients?

Need $\cos\theta_t = \cos\left(\frac{\pi}{2} + ib\right)$
$= -i\sinh(b)$

pure imaginary

So $r_\perp = \dfrac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$

$\rightarrow \dfrac{n_i\cos\theta_i - in_t\sinh b}{n_i\cos\theta_i + in_t\sinh b}$ complex!

Still have $E_{r0} = rE_{i0}$
complex r: phase shift between $\mathbf{E}_{\mathrm{inc}}$ and $\mathbf{E}_{\mathrm{refl}}$

Both $r_\perp$ and $r_\parallel$ have form

$$r = \frac{u+iv}{u-iv}$$

$r_\perp : u = n_i\cos\theta_i$ and $v = n_t\sinh b$
$r_\parallel : u = n_t\cos\theta_i$ and $v = n_i\sinh b$

If $z = u + iv$, then $r = z/z^*$

So $|r| = \dfrac{|z|}{|z^*|} = 1$: all light reflected

Write $z = |z|e^{i\phi}$, then $r = e^{2i\phi}$

Reflection phase shift: $2\phi = 2\tan^{-1}\left(\frac{v}{u}\right)$

To calculate numerically, just use good calculator (or computer program):

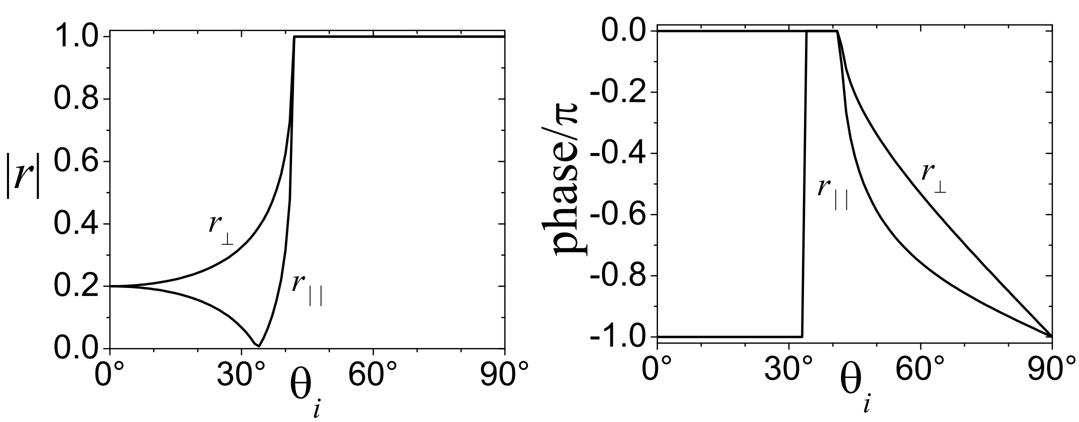
Evaluate

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

for $\theta_t = \sin^{-1}\left(\frac{n_i}{n_t}\sin\theta_i\right)$

Let computer deal with complex math

Plot for glass $\rightarrow$ air:



Use TIR to make good mirror
   be aware of phase shifts

For TIR, $R = |r|^2 = 1$
so all power reflected

But also have

$$t_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\parallel = \frac{2n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i}$$

do not equal zero!?

Have a transmitted field ($t \neq 0$)
but doesn't carry any energy ($R = 1$)

Question: How might this contradiction be resolved?

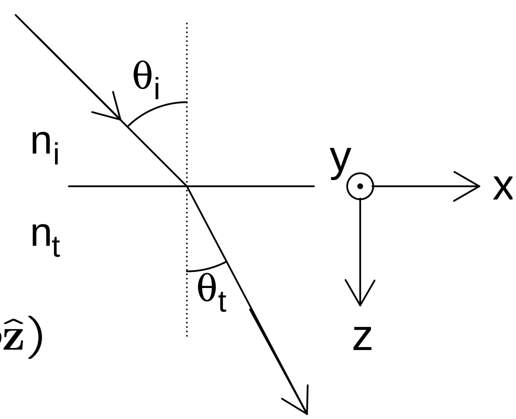
To see:

Transmitted wave: $\mathbf{E}_t = \mathbf{E}_{t0} e^{i(\mathbf{k}_t \cdot \mathbf{r} - \omega t)}$

with $\mathbf{k}_t = |k_t|(\sin\theta_t \hat{\mathbf{x}} + \cos\theta_t \hat{\mathbf{z}})$

For TIR, $\theta_t = \frac{\pi}{2} + b$
$\sin\theta_t \to \cosh b$
$\cos\theta_t \to i \sinh b$

So $\mathbf{k}_t \to k_t(\cosh b \hat{\mathbf{x}} + i \sinh b \hat{\mathbf{z}})$



Transmitted field

$$\mathbf{E}_t \to \mathbf{E}_{t0} e^{i[k_t(x \cosh b + iz \sinh b) - \omega t]}$$

$$= \mathbf{E}_{t0} e^{-k_t z \sinh b}\, e^{i(k_t x \cosh b - \omega t)}$$

Wave propagates in x direction

Decays exponentially in z direction

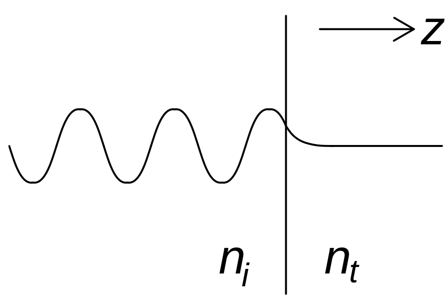
- Carries no energy away from surface

Called *evanenscent wave*

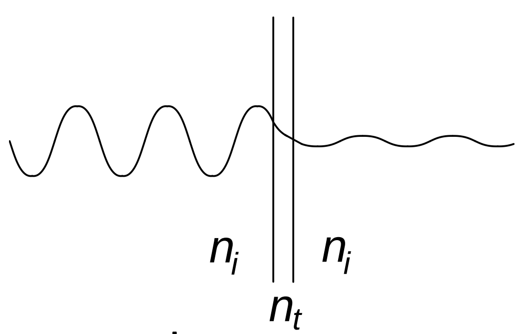
(Not same as exponential decay from absorption!)

Evanescent wave can be observed

Single surface:



Introduce second surface:



Transmitted wave appears!

Get transmission from tail of evanescent wave

For gap d, amplitude of transmitted wave
$\approx e^{-k_t d \sinh b}$

- reflection $\rightarrow 0$ smoothly as $d \rightarrow 0$

Called *frustrated total internal reflection*

Completely analogous to tunneling in QM

Still hope $T = 0$ for plain TIR

Should have defined $T = \text{Re}\left[\frac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] |t|^2$

In TIR, $\cos\theta_t = -i \sinh b$
real part = 0

So $T = 0$, as needed

Raises question: what if n_i or n_t is complex?

Reflection from metals (Hecht 4.8)

Saw previously that in absorbing medium

$$n \to n + i\frac{\alpha}{2k_0}$$

α = absorption coefficient

Get $\mathbf{E} = \mathbf{E}_0 e^{i(n\mathbf{k}_0\cdot\mathbf{r}-\omega t)}$

$$\to \mathbf{E}_0 e^{-\alpha\widehat{\mathbf{k}}\cdot\mathbf{r}/2}\, e^{i(n\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

and $I \propto |E_0|^2 \propto e^{-\alpha\widehat{\mathbf{k}}\cdot\mathbf{r}}$

Wave attenuates as it propagates

Question: How could you distinguish an evanescent wave from TIR and a plane wave exponentially decaying due to absorption? (Supposing no knowledge about the media.)

Normally don't want optical material to absorb

Important exception: mirrors
- light doesn't penetrate medium
- not much loss

How to apply Fresnel relations?
- Typically n_i, θ_i real
- n_t complex

Same equations apply

- Snell's law: $\sin\theta_t = \dfrac{n_i}{n_t}\sin\theta_i$

  (so θ_t complex)

- Use $\cos\theta_t = \sqrt{1-\sin^2\theta_t}$ as before

- Plug into equations for $r_\perp$, $r_\parallel$
  Get complex result

Hard to do by hand; easy on computer

Consider a very good absorber $\alpha \to \infty$

Then $n_i \sin\theta_i = \left(n_t + i\dfrac{\alpha}{2k_0}\right)\sin\theta_t$

means $\theta_t \to 0$

and $$r_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$\to \frac{n_i\cos\theta_i - n_t - i\alpha/2k_0}{n_i\cos\theta_i + n_t + i\alpha/2k_0}$$

$$\to -1$$

Similarly $r_\parallel \to 1$

So perfect absorber = perfect reflector

To make a true absorber:
need moderate α + porous surface
    reflected waves bounce many times
Example: soot (= carbon dust)

Use high-α material for mirror

Highest α: metals

Homework problem 5 from assignment 1:
    In conductive medium get current $\mathbf{J} = \sigma \mathbf{E}$
        σ = conductivity

Got $k = \sqrt{\epsilon \mu_0 \omega^2 + i\omega\mu_0\sigma}$

Rewrite

$$k = k_0 \sqrt{\frac{\epsilon}{\epsilon_0} + i\frac{\sigma}{\epsilon_0 \omega}}$$

Good conductor: silver
    $\sigma \approx 6 \times 10^7$ (Ω m)$^{-1}$ (at dc)

If $\lambda = 500$ nm and $\epsilon \approx \epsilon_0$

$$\frac{\sigma}{\epsilon_0 \omega} \approx 2000$$

So $k \approx k_0 \sqrt{2000i}$

$$\approx 45 k_0 \frac{1+i}{\sqrt{2}} = 30 k_0 (1+i)$$

Then expect $n \approx \alpha / 2k_0 \approx 30$

Actually, not that good at optical freqs

find $n = 0.3$ and $\alpha / 2k_0 = 4$

Still get $R \approx 0.95$ across visible

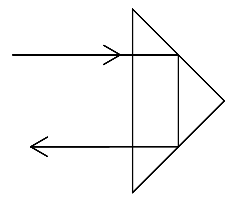
Practical notes on mirrors

- Typical metals:
  - Silver: $R \approx 0.95$ in visible, NIR
    - oxidizes quickly in air
  - Gold: $R \approx 0.95$ in NIR
    - doesn't oxidize
  - Aluminum: $R \approx 0.85$ in visible, NIR
    - oxidizes but easy to protect (SiO)
- Metals don't have Brewster angle

  typically dip in $R_{\|}$, but not to zero
- Wave penetrates fraction of λ

  typical 50-100 nm

Get better mirrors using dielectric layers
- discuss theory later
- can get $R = 0.99$ easily, 0.99999 with effort
- more expensive than metal

Could use TIR: 

Drawbacks:
- Reflection losses from first surface
- Beam displacement inconvenient
- Limited range of θ_i

Usually use when displacement desired

Summary

- Get TIR with internal incidence, $\theta_i > \theta_c$
- Perfect reflection, with phase shift
- Evanescent wave at surface
- For TIR or absorbing media,
Fresnel equations are complex
- Highly absorbing medium $\rightarrow$ good mirror