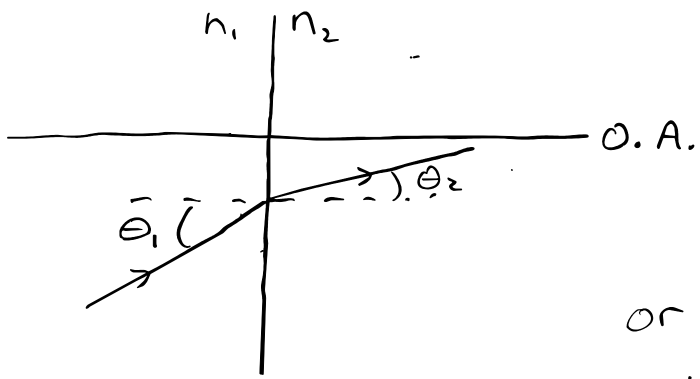
Phys 532/822 HW 1 Solutions

1. a)

(3) Dielectric interface



Clearly $r_{in} = r_{out}$

From Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

or $n_1 \theta_1 = n_2 \theta_2$

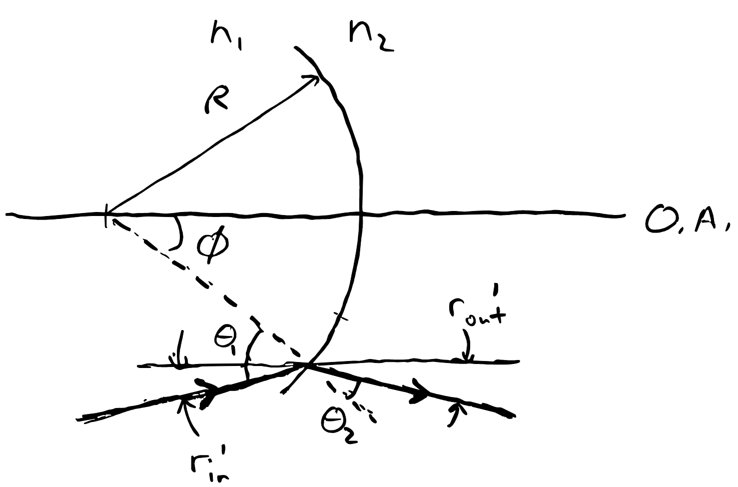
in paraxial limit

But $\theta_1 = r'_{in}$

$\theta_2 = r'_{out}$, so $r'_{out} = \frac{n_1}{n_2} r'_{in}$.

and $M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$

(4) Spherical interface



(Note $\phi < 0$ as drawn)

Again, $r_{in} = r_{out}$

Still have

$$\theta_2 = \frac{n_1}{n_2} \theta_1$$

but now

$$r'_{in} = \theta_1 + \phi$$

$$r'_{out} = \theta_2 + \phi$$

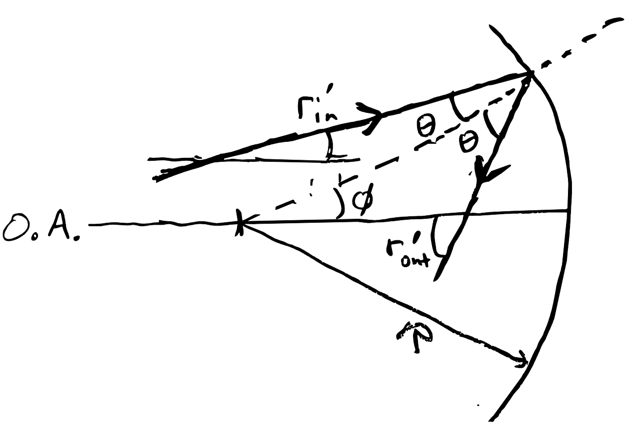
1. (cont)

and $\phi = \dfrac{r}{R}$

So, $$r'_{out} = \frac{n_1}{n_2}\Theta_1 + \phi$$
$$= \frac{n_1}{n_2}(r'_{in} - \phi) + \phi$$
$$= \frac{n_1}{n_2} r'_{in} + \left(1 - \frac{n_1}{n_2}\right)\frac{r}{R}$$

and matrix is $$M = \begin{bmatrix} 1 & 0 \\ \left(1-\frac{n_1}{n_2}\right)\frac{1}{R} & \frac{n_1}{n_2} \end{bmatrix}$$

(5) Spherical mirror



Again $r_{in} = r_{out}$,

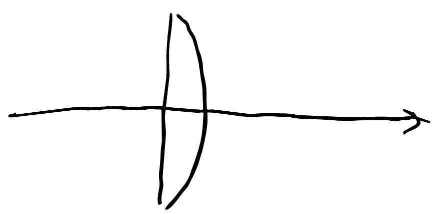
and
$$r'_{in} = \phi - \Theta$$
$$r'_{out} = -\phi - \Theta$$

So
$$r'_{out} = -\phi + r'_{in} - \phi$$
$$= r'_{in} - 2\phi$$
$$= r'_{in} - \frac{2r}{R}$$

Matrix is $$\begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix}$$

1.(b) Assume a thin lens, since thickness not given

Then $M_{TOT} = \begin{bmatrix} 1 & 0 \\ \frac{1-n}{R} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$



$$= \begin{bmatrix} 1 & 0 \\ \frac{1-n}{R} & 1 \end{bmatrix}$$

has form $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

for $\boxed{f = \frac{R}{n-1}}$

2. Note that $AD - BC$ is the determinant of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Also, $\det(M_1 M_2 \ldots M_N) = \det(M_1)\det(M_2) \ldots \det(M_N)$

For mirrors, $\det M_{mirror} = 1$

For dielectric interfaces, $\det M_{interface} = \frac{n_1}{n_2}$

For free propagation, $\det M_{free} = 1$

2. (cont)

So, for series of mirrors and interfaces,

$$\det M_{TOT} = \det M_{interface\,1} \cdot \det M_{int\,2} \cdot \ldots \cdot \det M_{int\,N}$$

$$= \frac{n_1}{n_2} \cdot \frac{n_2}{n_3} \cdots \frac{n_{N-1}}{n_N}$$

$$= \frac{n_1}{n_N}$$

or in language of problem, $\det M_{TOT} = \dfrac{n_1}{n_2}$

$$= AD - BC$$

3. Want a reentrant cavity

From class, round trip matrix is

$$\begin{bmatrix} 1 - \frac{2d}{R} & 2d - \frac{2d^2}{R} \\ \frac{4(d-R)}{R^2} & \frac{4d^2}{R^2} - \frac{6d}{R} + 1 \end{bmatrix}$$

So

$$b = \frac{A+D}{2} = \frac{1}{2}\left(2 - \frac{8d}{R} + \frac{4d^2}{R^2}\right) = 1 - \frac{4d}{R} + \frac{2d^2}{R^2} = \cos\theta$$

For rentrant cavity, want $m\theta = 2\pi l$

m = number of passes before repeating = 12

3. (cont)

ℓ can be any integer, but if ℓ and m have common factors, pattern will repeat before 12 passes

(Since then $\left(\frac{m}{k}\right)\Theta = 2\pi\left(\frac{\ell}{k}\right)$ for common factor k)

So, can have $\ell = 1, 5, 7, 11$

$$\Theta: \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

note $\frac{7\pi}{6} = -\frac{5\pi}{6}$ and $\frac{11\pi}{6} = -\frac{\pi}{6}$

Since $\cos(-\Theta) = \cos\Theta$, can ignore them

So check possible values of R:

In general

$$1 - \frac{4d}{R} + \frac{2d^2}{R^2} = b$$

$$2\left(\frac{d}{R}\right)^2 - 4\left(\frac{d}{R}\right) + 1 - b = 0$$

$$\frac{d}{R} = \frac{1}{4}\left(4 \pm \sqrt{16 - 8(1-b)}\right) = 1 \pm \sqrt{\frac{1+b}{2}}$$

$$R = \frac{d}{1 \pm \sqrt{\frac{1+b}{2}}}$$

3. (cont)

So, $\Theta = \frac{\pi}{6} \Rightarrow b = \frac{\sqrt{3}}{2}$

$$R = \frac{d}{1 \pm \sqrt{\frac{1+\sqrt{3}/2}{2}}} = \frac{d}{1 \pm 0.965} = 0.509\,d \text{ or } 28.6\,d$$

or, $\Theta = \frac{5\pi}{6}, \quad b = -\frac{\sqrt{3}}{2}$

$$R = \frac{d}{1 \pm \sqrt{\frac{1-\sqrt{3}/2}{2}}} = \frac{d}{1 \pm .067} = 0.937\,d \text{ or } 1.072\,d$$

Smallest possible R is $0.509\,d$

$$\boxed{R = 15.3\text{ cm}}$$

4. Calculate matrix, starting from R_4: (in mm)

$$\begin{bmatrix} 1 & 570 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{100} & 1 \end{bmatrix} \begin{bmatrix} 1 & 130 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{100} & 1 \end{bmatrix} \begin{bmatrix} 1 & 220 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{150} & 1 \end{bmatrix} \begin{bmatrix} 1 & 660 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_4} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.296 & 566.16 \\ -0.0019 & -0.192 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_4} & 1 \end{bmatrix}$$

$$\equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R_4} & 1 \end{bmatrix} = \begin{bmatrix} a - \frac{2b}{R_4} & b \\ c - \frac{2d}{R_4} & d \end{bmatrix}$$

4. (cont)

Cavity is stable if $\left| \frac{A+D}{2} \right| \leq 1$

$$\frac{1}{2}\left| a+d-\frac{2b}{R_4} \right| \leq 1$$

$$-2 \leq a+d-\frac{2b}{R_4} \leq 2$$

$$-2-a-d \leq -\frac{2b}{R_4} \leq 2-a-d$$

$$2+a+d \geq \frac{2b}{R_4} \geq a+d-2$$

$$\frac{2b}{2+a+d} \leq R_4 \leq \frac{2b}{a+d-2}$$

$a+d = 0.104$

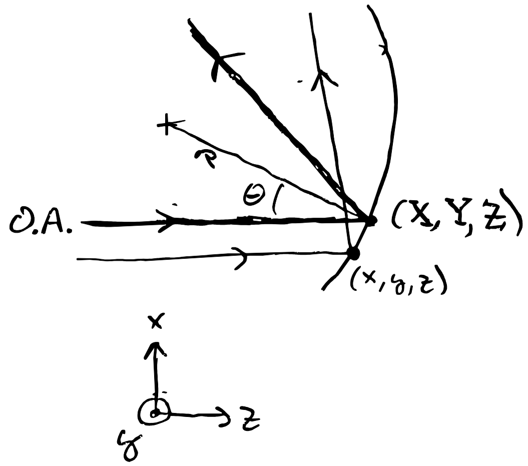
$$\frac{1132}{2.104} \leq R_4 \leq \frac{-1132}{1.896}$$

$$538\text{ mm} \leq R_4 \leq -597\text{ mm}$$

So, need either $\boxed{R_4 \geq 538\text{ mm}}$ or $\boxed{R_4 \leq -597\text{ mm}}$

In real laser, $R_4 = \infty$ (flat)

6. Set up coordinates, origin at center of sphere



Only care about incident rays with $r_{in}' = 0$, so take

$\hat{v}_{in} = \hat{z}$

Label point where optic axis intersects sphere as $(X, 0, Z_1)$, $Z_1 = \sqrt{R^2 - X^2}$

and point where ray intersects sphere as (x, y, z)

with $z = \sqrt{R^2 - x^2 - y^2}$

and normal to surface is

$$\hat{n} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{R} \quad \text{for ray}$$

and

$$\hat{n}_o = \frac{X\hat{x} + Z\hat{z}}{R} \quad \text{for axis}$$

Define $\hat{a}_{in}$ = input vector for axis $= \hat{z}$

$\hat{a}_{out}$ = output vector for axis

Then

$$\hat{a}_{out} = \hat{z} - 2(\hat{n}_o \cdot \hat{z})\hat{n}_o$$

$$\hat{v}_{out} = \hat{z} - 2(\hat{n} \cdot \hat{z})\hat{n}$$

6. (cont)

For M_{11}, take $x = X + dx$, $y = 0$

$$\text{so} \quad z = \sqrt{R^2 - (X+dx)^2} = \sqrt{R^2 - X^2 - 2X\,dx}$$
$$= \sqrt{Z_1^2 - 2X\,dx}$$
$$\approx Z\left(1 - \frac{X}{Z_1^2}\,dx\right)$$
$$z = Z_1 - \frac{X}{Z_1}\,dx$$
$$= Z_1 - \tan\theta\,dx$$

So $$\hat{n} = \frac{(X + dx)\hat{x} + (Z_1 - \tan\theta\,dx)\hat{z}}{R}$$

$$= \hat{n}_0 + (\hat{x} - \tan\theta\,\hat{z})\frac{dx}{R} \equiv \hat{n}_0 + d\vec{n}$$

Then
$$\hat{v}_{out} = \hat{z} - 2\left[(\hat{n}_0 + d\vec{n})\cdot\hat{z}\right](\hat{n}_0 + d\vec{n})$$
$$= \underbrace{\hat{z} - 2(\hat{n}_0\cdot\hat{z})\,\hat{n}_0}_{\hat{a}_{out}} - 2(d\vec{n}\cdot\hat{z})\hat{n}_0 - 2(\hat{n}_0\cdot\hat{z})\,d\vec{n} \quad *$$

$$= \hat{a}_{out} - 2\left(-\tan\theta\,\frac{dx}{R}\right)\frac{X\hat{x} + Z\hat{z}}{R} - 2\,\frac{Z}{R}\,(\hat{x} - \tan\theta\,\hat{z})\frac{dx}{R}$$

$$\hat{v}_{out} - \hat{a} = 2\,\frac{dx}{R}\left[\left(\tan\theta\,\frac{X}{R} - \frac{Z_1}{R}\right)\hat{x} + 2\tan\theta\,\frac{Z_1}{R}\,\hat{z}\right]$$

$$\text{but} \quad \frac{X}{R} = \sin\theta \qquad \frac{Z}{R} = \cos\theta$$

$$= 2\,\frac{dx}{R}\left[(\tan\theta\sin\theta - \cos\theta)\,\hat{x} + 2\sin\theta\,\hat{z}\right]$$

6. (cont)

$$|\hat{v}_{out} - \hat{a}| = 2\frac{dx}{R}\sqrt{(\tan\theta\sin\theta - \cos\theta)^2 + 4\sin^2\theta}$$

$$= 2\frac{dx}{R}\sqrt{\tan^2\theta\sin^2\theta - 2\sin^2\theta + \cos^2\theta + 4\sin^2\theta}$$

$$= 2\frac{dx}{R}\sqrt{\frac{\sin^4\theta}{\cos^2\theta} + 2\sin^2\theta + \cos^2\theta}$$

$$= 2\frac{dx}{R}\sqrt{\frac{(\sin^2\theta + \cos^2\theta)^2}{\cos^2\theta}} = dx\,\frac{2}{R\cos\theta}$$

but $|\hat{v}_{out} - \hat{a}| = |r_{out}'|$, so $|C| = \frac{2}{R\cos\theta}$

Can get sign from behavior at $\theta = 0$, so

$$C_{\parallel} = -\frac{2}{R\cos\theta}$$

For $M_\perp$, $x = X$, $y = dy$

$$z = \sqrt{R^2 - X^2 - dy^2} = \sqrt{Z_1^2 - dy^2} = Z_1$$

$$\hat{n} = \frac{X\hat{x} + dy\,\hat{y} + Z_1\hat{z}}{R} = \hat{n}_0 + \frac{dy}{R}\hat{y}$$

So now $d\vec{n} = \frac{dy}{R}\hat{y}$

and from *

$$\hat{v}_{out} = \hat{a}_{out} - 2(\hat{n}_0\cdot\hat{z})\,d\vec{n}$$

$$= \hat{a}_{out} - 2\cos\theta\,\frac{dy}{R}\hat{y}$$

6. (cont)

Can see immediately that

$$r'_{out} = -\frac{2\cos\theta}{R}\, r_{in}$$

so

$$\boxed{C_\perp = -\frac{2\cos\theta}{R}}$$