Phys 532 Assignment 10 Solutions

1. a) $V = V_0\left(\frac{1}{2}\frac{x^2}{a^2} + \frac{1}{3}\frac{x^3}{a^3}\right)$

force $F = -\frac{dV}{dx} - eE_0 e^{i\omega t} = m\ddot{x}$

So $m\ddot{x} + \frac{V_0}{a^2}x + \frac{V_0}{a^3}x^2 = -eE_0 e^{i\omega t}$

$$\boxed{\ddot{x} + \frac{V_0}{ma^2}x + \frac{V_0}{ma^3}x^2 = -\frac{eE_0}{m}e^{i\omega t}}$$

For small x, neglect x^2 term

$$\ddot{x} + \frac{V_0}{ma^2}x = -\frac{eE_0}{m}e^{i\omega t}$$

Has form $\ddot{x} + \omega_0^2 x = \dots$

with $\boxed{\omega_0 = \sqrt{\frac{V_0}{ma^2}}}$

b) Substitute into equation of motion

$$-\omega^2 x_1 e^{i\omega t} - 4\omega^2 x_2 e^{2i\omega t} - 9\omega^2 x_3 e^{3i\omega t} + \dots$$

$$+ \frac{V_0}{ma^2}\left(x_1 e^{i\omega t} + x_2 e^{2i\omega t} + x_3 e^{3i\omega t} + \dots\right)$$

$$+ \frac{V_0}{ma^3}\left(x_1 e^{i\omega t} + x_2 e^{2i\omega t} + x_3 e^{3i\omega t} + \dots\right)^2$$

$$= -\frac{eE_0}{m}e^{i\omega t}$$

Note

$$x(t)^2 = x_1^2 e^{2i\omega t} + 2x_1x_2 e^{3i\omega t} + \ldots$$

only need terms up to 3ω

So equation is

$$\left(-\omega^2 x_1 + \frac{V_0}{ma^2} x_1 + \frac{eE_0}{m}\right) e^{i\omega t}$$

$$+ \left(-4\omega^2 x_2 + \frac{V_0}{ma^2} x_2 + \frac{V_0}{ma^3} x_1^2\right) e^{2i\omega t}$$

$$+ \left(-9\omega^2 x_3 + \frac{V_0}{ma^2} x_3 + \frac{2V_0}{ma^3} x_1 x_2\right) e^{3i\omega t} + \ldots = 0$$

Coefficient of each exponential must vanish,

so

$$\boxed{x_1 = -\frac{eE_0}{m} \frac{1}{\frac{V_0}{ma^2} - \omega^2}} = -\frac{eE_0}{m} \frac{1}{\omega_0^2 - \omega^2}$$

$$x_2 = -\frac{V_0}{ma^3} x_1^2 \frac{1}{\frac{V_0}{ma^2} - 4\omega^2}$$

$$\boxed{x_2 = -\frac{V_0}{ma^3} \frac{e^2E_0^2}{m^2} \frac{1}{\left(\frac{V_0}{ma^2} - \omega^2\right)^2 \left(\frac{V_0}{ma^2} - 4\omega^2\right)}}$$

$$= -\frac{e^2E_0^2}{m^2 a} \frac{\omega_0^2}{(\omega_0^2 - \omega^2)^2 (\omega_0^2 - 4\omega^2)}$$

$$X_3 = -\frac{2V_0}{ma^3} x_1 x_2 \frac{1}{\frac{V_0}{ma^2} - 9\omega^2}$$

$$\boxed{x_3 = -\frac{2}{a^2}\left(\frac{eE_0}{m}\right)^3 \frac{\omega_0^4}{(\omega_0^2-\omega^2)^3(\omega_0^2-4\omega^2)(\omega_0^2-9\omega^2)}}$$

c) In limit $\omega \ll \omega_0$

$$x_1 \to -\frac{eE_0}{m}\frac{1}{\omega_0^2} = -\frac{eE_0}{m}\frac{ma^2}{V_0} = -\frac{ea^2E_0}{V_0}$$

So $\boxed{\frac{x_1}{a} \to -\frac{eaE_0}{V_0}}$

and

$$x_2 \to -\frac{V_0}{ma^3} x_1^2 \frac{1}{\omega_0^2} = -\frac{1}{a} x_1^2$$

so $\boxed{\frac{x_2}{x_1} \to -\frac{x_1}{a}}$

and $x_3 \to -\frac{2V_0}{ma^3} x_1 x_2 \frac{1}{\omega_0^2} = -\frac{2x_1x_2}{a}$

so $\boxed{\frac{x_3}{x_2} \to -\frac{2x_1}{a}}$

So if $eE_0 \ll \frac{V_0}{a}$, then $\frac{x_1}{a} \ll 1$ and terms are decreasing.

2. a)

$$P(t) = 2d\, E(t)^2$$

$$= 2d \left[E^{\omega_1} e^{i\omega_1 t} + E^{\omega_2} e^{i\omega_2 t} + E^{\omega_1 *} e^{-i\omega_1 t} + E^{\omega_2 *} e^{-i\omega_2 t} \right]^2 \frac{1}{4}$$

$$= \frac{d}{2} \Big[(E^{\omega_1})^2 e^{2i\omega_1 t} + (E^{\omega_2})^2 e^{2i\omega_2 t} + (E^{\omega_1 *})^2 e^{-2i\omega_1 t} + (E^{\omega_2 *})^2 e^{-2i\omega_2 t}$$

$$+ 2E^{\omega_1} E^{\omega_2} e^{i(\omega_1+\omega_2)t} + 2|E^{\omega_1}|^2 + 2E^{\omega_1} E^{\omega_2 *} e^{i(\omega_1-\omega_2)t}$$

$$+ 2E^{\omega_2} E^{\omega_1 *} e^{i(\omega_2-\omega_1)t} + 2|E^{\omega_2}|^2 + 2E^{\omega_1 *} E^{\omega_2 *} e^{-i(\omega_1+\omega_2)t} \Big]$$

Only care about terms oscillating at $\omega_3 = \omega_1 + \omega_2$:

$$\frac{1}{2}\left(P^{\omega_3} e^{i\omega_3 t} + P^{\omega_3 *} e^{-i\omega_3 t} \right) = \frac{d}{2}\Big[2E^{\omega_1} E^{\omega_2} e^{i(\omega_1+\omega_2)t}$$

$$+ 2E^{\omega_1 *} E^{\omega_2 *} e^{-i(\omega_1+\omega_2)t} \Big]$$

So $\boxed{P^{\omega_3} = 2d\, E^{\omega_1} E^{\omega_2}}$

b) If $\omega_1 = \omega_2 + \varepsilon$, then for $t \ll \frac{1}{\varepsilon}$, can't distinguish $\omega_1 + \omega_2$, $2\omega_2$, and $2\omega_1$

$\rightarrow$ all look like ω_3

So, $$\frac{1}{2} P^{\omega_3} e^{i\omega_3 t} = \frac{d}{2}\Big[(E^{\omega_1})^2 e^{2i\omega_1 t} + (E^{\omega_2})^2 e^{2i\omega_2 t}$$

$$+ 2E^{\omega_1} E^{\omega_2} e^{i(\omega_1+\omega_2)t} \Big]$$

$$\frac{1}{2} P^{\omega_3} = \frac{d}{2}\left(E^{\omega_1} + E^{\omega_2}\right)^2$$

$$\boxed{P^{\omega_3} = d\left(E^{\omega_1} + E^{\omega_2}\right)^2 \approx d\left(E^{\omega}\right)^2}$$

3. For point group symmetry 32, d_{ij} tensor looks like

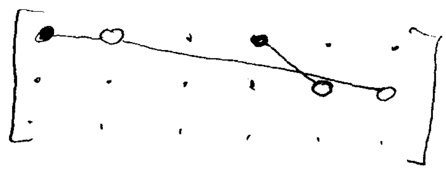


Table 16.2 gives $d_{11} = 517 \times \frac{1}{9} \times 10^{-22} \frac{C}{V^2} = 5.7 \times 10^{-21} \frac{C}{V^2}$

and $d_{12} = d_{xyy} = d_{26} = d_{yxy} = -d_{11}$

Also have $d_{14} = d_{xyz} = -d_{25} = -d_{yxz}$

[From Kleinman symmetry, expect
$d_{14} = d_{25}$
so can assume $d_{14} \approx 0$]

From Table, have indices

λ	n_o	n_e
5.3 μm	4.856	6.307
10.6 μm	4.794	6.243

Since $n_o(5.3)$ lies between n_o & n_e at $10.6\,\mu m$,

Want output beam to be ordinary,
input to be extraordinary

Phase matching angle:

Want $$\frac{1}{n_o(5.3)^2} = \frac{\cos^2\theta}{n_o(10.6)^2} + \frac{\sin^2\theta}{n_e(10.6)^2}$$

where θ: angle between propagation direction and z-axis

write $$\frac{1}{(n_o^{2\omega})^2} = \frac{1-\sin^2\theta}{(n_o^{\omega})^2} + \frac{\sin^2\theta}{(n_e^{\omega})^2}$$

$$\frac{1}{(n_o^{2\omega})^2} - \frac{1}{(n_o^{\omega})^2} = \sin^2\theta\left(\frac{1}{(n_e^{\omega})^2} - \frac{1}{(n_o^{\omega})^2}\right)$$

$$\sin^2\theta = \frac{\frac{1}{(n_o^{2\omega})^2} - \frac{1}{(n_o^{\omega})^2}}{\frac{1}{(n_e^{\omega})^2} - \frac{1}{(n_o^{\omega})^2}}$$

$$= \frac{\frac{1}{(4.856)^2} - \frac{1}{(4.794)^2}}{\frac{1}{(6.243)^2} - \frac{1}{(4.794)^2}}$$

$$= 0.0618$$

$$\sin\theta = 0.249$$

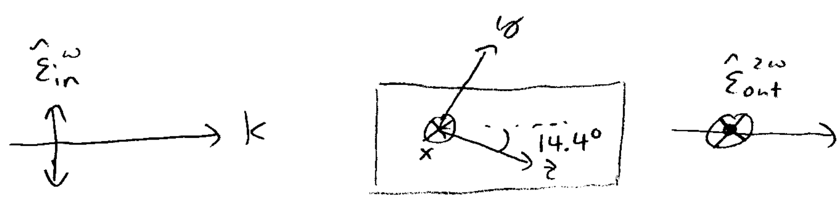
$$\boxed{\theta = 14.4^\circ}$$

So angle between $\vec{k}$ and z axis is $14.4°$

Since we have ordinary beam at 2ω, need to use d_{xjk} or d_{yjk}

Want orthogonal polarization at ω, so use $d_{xyy} = d_{12} = -517\,\varepsilon_0 \frac{pm}{V}$

So, orientation looks like



Then E_y^{ω} component is $E^{\omega}\cos\theta$

But $P^{2\omega} \propto (E_y^{\omega})^2$, so

$$d' = \cos^2\theta \, d_{12}$$

$$= 0.938\, d_{12}$$

$$\boxed{d' = 5.39\times10^{-21}\ \frac{C}{V^2}}$$

4. From Dmitriev, Gurzadyan & Nikogosyan, 3rd ed
pg 96

Symmetry class 3m, uniaxial

So from Yariv,

$$d_{22} = -d_{21} = -d_{16}$$
$$d_{24} = d_{15}$$
$$d_{31} = d_{32}$$
$$d_{33}$$

Kleinman symmetry gives $d_{24} = d_{yyz} = d_{zyy} = d_{32}$

Transmission range 200 nm to 3.5 μm

Coefficient values $d_{22} = 2.3 \frac{pm}{V}$

$d_{31} = 0.16 \frac{pm}{V}$

d_{33} not given
(unusable for phase matching)

Indices of refraction

λ	n_o	n_e
532 nm	1.6755	1.5549
1064 nm	1.6551	1.5425