Phys 532/822 Assignment 2 Solutions

1. Have

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}$$

$$q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2}$$

$$= \frac{A_2\left(\frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}\right) + B_2}{C_2\left(\frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}\right) + D_2}$$

$$= \frac{(A_1 A_2 + C_1 B_2) q_1 + (B_1 A_2 + D_1 B_2)}{(A_1 C_2 + C_1 D_2) q_1 + (B_1 C_2 + D_1 D_2)}$$

while

$$\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A_1 A_2 + C_1 B_2 & B_1 A_2 + D_1 B_2 \\ B_1 C_2 + C_1 D_2 & B_1 C_2 + D_1 D_2 \end{bmatrix}$$

$$\equiv \begin{bmatrix} A_T & B_T \\ C_T & D_T \end{bmatrix}$$

So, $$q_3 = \frac{A_T q_1 + B_T}{C_T q_1 + D_T}$$ as claimed.

2. Start with $q_1 = jz_0 = \dfrac{j\pi w_0^2}{\lambda}$ at $z = 0$

$$q_1 = j\,14.76\text{ mm}$$

at lens, $q_2 = 35\text{ mm} + j\,14.76\text{ mm}$

$$\frac{1}{q_2} = 0.0243\text{ mm}^{-1} - j\,0.0102\text{ mm}^{-1}$$

after lens, $\dfrac{1}{q_3} = \dfrac{1}{q_2} - \dfrac{1}{f}$

$$= -0.0157\text{ mm}^{-1} - j\,0.0102\text{ mm}^{-1}$$

$$q_3 = -44.7\text{ mm} + j\,29\text{ mm}$$

$$= z + jz_0$$

So,

focus is $\boxed{44.7\text{ mm after lens}}$

with $w_0 = \sqrt{\dfrac{\lambda z_0}{\pi}} = \boxed{70\,\mu\text{m}}$

In geometrical optics, with focus at $z=0$, would have image focus at

$$\frac{1}{S_i} = \frac{1}{f} - \frac{1}{S_o}$$

$$= \frac{1}{25\text{ mm}} - \frac{1}{35\text{ mm}} = \frac{1}{87.5\text{ mm}}$$

$$\boxed{S_i = 87.5\text{ mm}}$$

3. Know $I(r) = I_0 e^{-2r^2/w^2}$

$$= I_0 e^{-2x^2/w^2} e^{-2y^2/w^2}$$

So, transmitted fraction of power is

$$t(x_1) = \frac{\int_{-\infty}^{\infty} dy \int_{-\infty}^{x_1} dx \; I(x,y)}{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \; I(x,y)}$$

$$= \frac{\int_{-\infty}^{x_1} e^{-2x^2/w^2} dx}{\int_{-\infty}^{\infty} e^{-2x^2/w^2} dx}$$

$u = \sqrt{2}\,\frac{x}{w}$

$u_1 = \sqrt{2}\,\frac{x_1}{w}$

$$= \frac{\int_{-\infty}^{u_1} e^{-u^2} du}{\int_{-\infty}^{\infty} e^{-u^2} du}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{u_1} e^{-u^2} du$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \int_0^{u_1} e^{-u^2} du \right]$$

$$t(x_1) = \frac{1}{2} + \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_0^{u_1} e^{-u^2} du \right] = \frac{1}{2}\left[1 + \operatorname{erf}(u_1)\right]$$

3. (cont)

So, have $t(x_1) = 0.7 = \frac{1}{2}[1 + \mathrm{erf}(u_1)]$

$$\Rightarrow \mathrm{erf}(u_1) = 0.4$$

and $t(x_2) = 0.3 = \frac{1}{2}[1 + \mathrm{erf}(u_2)]$

$$\Rightarrow \mathrm{erf}(u_2) = -0.4$$

So, $u_1 = 0.37 = \sqrt{2}\,\frac{x_1}{w}$

$u_2 = -0.37 = \sqrt{2}\,\frac{x_2}{w}$

and $\Delta = x_1 - x_2 = \frac{w}{\sqrt{2}}(0.37 + 0.37)$

or $\boxed{w = 1.91\,\Delta}$

4. Have $q_0 = i z_0 = i\,\frac{\pi w_0^2}{\lambda}$

after lens, $\frac{1}{q_1} = -\frac{1}{f} - \frac{i}{z_0}$

$$q_1 = \frac{-\frac{1}{f} + \frac{i}{z_0}}{\frac{1}{f^2} + \frac{1}{z_0^2}} = \frac{-f z_0^2 + i z_0 f^2}{z_0^2 + f^2}$$

Want $\mathrm{Re}\, q_1 = -d$ to have focus at $z = d$

4. (cont)

So $$\frac{f z_0^2}{z_0^2 + f^2} = d$$

$$d f^2 - z_0^2 f + d z_0^2 = 0$$

$$f = \frac{1}{2d}\left(z_0^2 \pm \sqrt{z_0^4 - 4d^2 z_0^2}\right)$$

$$f = \frac{z_0}{2d}\left(z_0 \pm \sqrt{z_0^2 - 4d^2}\right)$$

$$\boxed{f = \frac{\pi w_0^2}{2\lambda d}\left[\frac{\pi w_0^2}{\lambda} \pm \sqrt{\frac{\pi^2 w_0^4}{\lambda^2} - 4d^2}\right]}$$

Are both solutions valid?

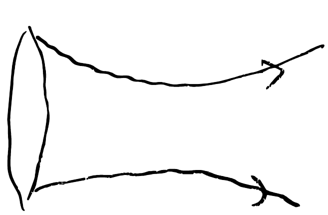
Check $z_0 \to \infty$ (ray optics limit):

$f_+ \to \infty$ valid, beam stays collimated

$$f_- \to \frac{z_0}{2d}\left[z_0 - z_0\left(1 - \frac{2d^2}{z_0}\right)\right] = d$$

valid, geometrical optics

So, generally two solutions:



and

4. (cont)

Largest possible d given by $4d^2 = z_0^2$

$$d = \frac{z_0}{2}$$

$$\boxed{d_{max} = \frac{\pi w_0^2}{2\lambda}}$$

5.(a) From last assigment, know round trip ray matrix starting from M_4 is (in mm) $\begin{bmatrix} 0.296 & 566.16 \\ -0.0019 & -0.192 \end{bmatrix}$.

(note that since M_4 is flat, don't need to specify which side of it we're on)

So, at M_4, $q_4 = \frac{1}{C}\left[\frac{A-D}{2} \pm i\sqrt{1-\left(\frac{A+D}{2}\right)^2}\right]$

$$= -131\text{ mm} + i\,535\text{ mm}$$

So for leg from M_1 to M_3, have focus 131 mm past M_4

(can treat as one leg since M_4 is flat)

Spot size $= \sqrt{\frac{\lambda z_0}{\pi}} = 0.36$ mm

5. (cont)

At M_3, $q_3 = 529\text{mm} + j\,535\text{mm}$,

$\therefore \frac{1}{q_3} = 9.35\times10^{-4} - j\,9.45\times10^{-4}$

So after M_3, $\frac{1}{q_3'} = -1.24\times10^{-2} - j\,9.45\times10^{-4}$

$q_3' = -80 + j\,6.11$

So focus between R_3 + R_2 is 80mm past R_3

with $w_0 = 39\mu m$

At R_2, $q_2 = 140 + j\,6.11$

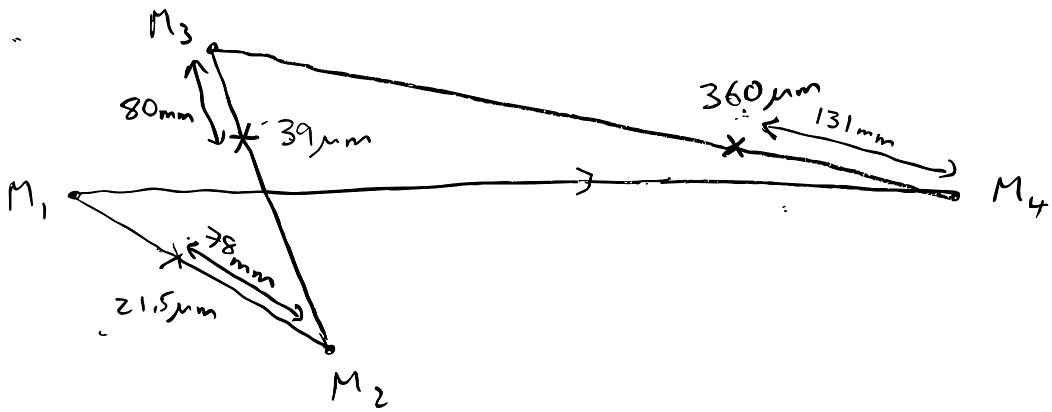
$\frac{1}{q_2} = 7.13\times10^{-3} - j\,3.11\times10^{-4}$

after R_2, $\frac{1}{q_2'} = -0.0129 - j\,3.11\times10^{-4}$

$q_2' = -77.6 + j\,1.877$

So, focus 77.6 mm past R_2, with $w_0 = 21.5\mu m$

So have:



b) Output beam will have character of leg from M_1 to M_3

Far field divergence angle determined by spot size at focus:

$$\theta = \frac{\lambda}{\pi w_0}$$

Here $w_0 = 0.36\,mm$

$$\boxed{\theta = 0.69\ mrad}$$

6. Have $\Delta\nu_L = \frac{c}{L}$ L = round trip length $= 1.58m$

So $\boxed{\Delta\nu_L = 190\ MHz}$

7.

$$\nabla_t^2 \psi - 2ik \frac{\partial \psi}{\partial z} = 0$$

if $\psi = f(x,y) N(z) e^{-\alpha(z) r^2}$

$$\frac{\partial \psi}{\partial z} = f(x,y) N' e^{-\alpha r^2} - \alpha' r^2 f N e^{-\alpha r^2}$$

$$= \left(\frac{N'}{N} - \alpha' r^2 \right) \psi$$

So,

$$\nabla_t^2 \psi - 2ik \left(\frac{N'}{N} - \alpha' r^2 \right) \psi = 0$$

$$\nabla_t^2 \psi + (2ik\alpha) r^2 \psi = \left(2ik \frac{N'}{N} \right) \psi$$

Has form of Schrodinger equation for harmonic oscillator

$$-\frac{\hbar^2}{2m} \nabla_t^2 \psi + \frac{1}{2} m\omega^2 r^2 \psi = E \psi$$

with $\dfrac{m^2\omega^2}{\hbar^2} \rightarrow -2ik\alpha$

$$\frac{2mE}{\hbar^2} \rightarrow -2ik \frac{N'}{N}$$