Phys 532 HW 3 Solutions

1. Know $\Delta\nu_{1/2} = \frac{\Gamma}{2\pi}\Delta\nu_L$

$$\Gamma = \alpha L - \ln(r_1 r_2 \ldots r_N)$$

additional cavity loss of 2% acts just like another mirror, so

$$\Gamma = -\ln\left[(0.995)^3(0.95)(0.98)\right]$$
$$= 0.0865$$

and $\Delta\nu_L = 190$ MHz (from previous assignment)

So, $\boxed{\Delta\nu_{1/2} = 2.6 \text{ MHz}}$

2. (a) With $M=0$, have

$$V_L = -L\dot{I} \Rightarrow \dot{V}_L = -L\ddot{I}$$
$$\dot{V}_c = \frac{1}{c}I$$
$$V_L - V_c = IR \Rightarrow \dot{V}_L - \dot{V}_c = \dot{I}R$$

Combine to eliminate V's:

$$-L\ddot{I} - \frac{1}{c}I = \dot{I}R$$

or

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = 0$$

2. (cont)

$$\ddot{I} + \gamma \dot{I} + \omega_0^2 I = 0$$

Try $I = e^{\lambda t}$!

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0$$

$$\lambda = \frac{1}{2}\left(-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)$$

If $\gamma \ll \omega_0$, then $\lambda = -\frac{\gamma}{2} \pm i\omega_0$

So, $I(t) = e^{-\gamma t/2}\left(A e^{i\omega_0 t} + B e^{-i\omega_0 t}\right)$

If $I(0) = I_1$ then $A + B = I_1$

$$\left.\frac{dI}{dt}\right|_0 = -\frac{\gamma}{2}(A+B) + i\omega_0(A-B) = 0$$

For $\gamma \ll \omega_0$, just take $A = B$,

so $\boxed{I(t) = I_1 e^{-\gamma t/2} \cos\omega_0 t}$

b) $V_c = \frac{1}{C}\int_0^t I(t')\,dt'$

$$= \frac{1}{C}\frac{I_1}{2}\int_0^t e^{(-\frac{\gamma}{2}+i\omega_0)t'} + e^{(-\frac{\gamma}{2}-i\omega_0)t'}\,dt'$$

$$= \frac{I_1}{2C}\left[\frac{e^{(-\frac{\gamma}{2}+i\omega_0)t'}}{-\frac{\gamma}{2}+i\omega_0} + \frac{e^{(-\frac{\gamma}{2}-i\omega_0)t'}}{-\frac{\gamma}{2}-i\omega_0}\right]\Bigg|_0^t$$

For $\gamma \ll \omega_0$,

$$V_c(t) = \frac{I_1}{2C}\frac{1}{i\omega_0}\left[e^{-\frac{\gamma t}{2}}(2i \sin \omega_0 t)\right]\Big|_0^t$$

$$= \frac{I_1}{\omega_0 C} e^{-\gamma t/2} \sin \omega_0 t$$

So, $\mathcal{E} = \frac{1}{2} C V_c^2 + \frac{1}{2} L I^2$

$$= \frac{1}{2}\frac{I_1^2}{\omega_0^2 C} e^{-\gamma t} \sin^2 \omega_0 t + \frac{1}{2} L I_1^2 e^{-\gamma t} \cos^2 \omega_0 t$$

with $\omega_0^2 = \frac{1}{LC}$

$$= \frac{1}{2} L I_1^2 e^{-\gamma t}(\sin^2 \omega_0 t + \cos^2 \omega_0 t)$$

$$\mathcal{E} = \frac{1}{2} L I_1^2 e^{-\gamma t}$$

So, $\frac{d\mathcal{E}}{dt} = -\gamma \frac{1}{2} L I_1^2 e^{-\gamma t} = -\gamma \mathcal{E}$

and $\boxed{Q = \frac{\omega_0}{\gamma}}$

2. (cont)

c) For $M \neq 0$, just want steady state solution for $V_0 = Ae^{i\omega t}$

so take

$$I_0(t) = I_0 e^{i\omega t} \qquad V_1(t) = V_1 e^{i\omega t}$$

$$I(t) = I e^{i\omega t} \qquad \text{etc}$$

Then

(1) $V_0 - V_1 = I_0 R_0$

(2) $V_1 = L_0 \dot{I}_0 + M\dot{I} \Rightarrow V_1 = i\omega L_0 I_0 + i\omega M I$

(3) $V_L = -L\dot{I} - M\dot{I}_0 \Rightarrow V_L = -i\omega L I - i\omega M I_0$

(4) $V_L - V_C = IR$

(5) $\dot{V}_C = \frac{I}{C} \Rightarrow i\omega V_C = \frac{1}{C} I$

Solve for I:

(1) $V_1 = V_0 - I_0 R_0$

then

(2) $V_0 - I_0 R_0 = i\omega L_0 I_0 + i\omega M I$

$$I_0 (R_0 + i\omega L_0) = V_0 + i\omega M I$$

Can take $R_0 \gg i\omega L_0$, so

$$I_0 = \frac{1}{R_0}(V_0 + i\omega M I)$$

2. (cont)

Then from (3)

$$V_L = -i\omega L I - i\omega M \frac{1}{R_0}(V_0 + i\omega M I)$$

(4)

$$V_C = V_L - IR$$

$$= -i\omega L I - \frac{i\omega M}{R_0}(V_0 + i\omega M I) - IR$$

$$= I\left[-i\omega L - R + \frac{\omega^2 M^2}{R_0}\right] - \frac{i\omega M}{R_0} V_0$$

But we have $\omega^2 M^2 \ll R R_0$, so

$$\frac{\omega^2 M^2}{R_0} \ll R \text{, and}$$

$$V_C \approx I[-i\omega L - R] - \frac{i\omega M}{R_0} V_0$$

and from (5):

$$i\omega\left\{ I(-i\omega L - R) - \frac{i\omega M}{R_0} V_0 \right\} = \frac{1}{C} I$$

$$\frac{\omega^2 M}{R_0} V_0 = I\left[\frac{1}{C} - \omega^2 L + i\omega R\right]$$

$$= I L\left[\frac{1}{LC} - \omega^2 + i\omega \frac{R}{L}\right]$$

2. (cont)

$$\frac{\omega^2 M}{R_0} V_0 = I L\left(\omega_0^2 - \omega^2 + i\omega\gamma\right)$$

or $$\boxed{I = \frac{\omega^2 M}{R_0 L} \frac{V_0}{\omega_0^2 - \omega^2 + i\omega\gamma}}$$ here $V_0 = A$

Simplify for $\omega \approx \omega_0$:

$$\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$$

$$\approx 2\omega(\omega_0 - \omega)$$

so $$\boxed{I \approx \frac{\omega_0 M}{2R_0 L} \frac{A}{\omega_0 - \omega + i\frac{\gamma}{2}}}$$

Energy in oscillator $\mathcal{E} \propto |I|^2$

$$\propto \frac{1}{\left|\omega_0 - \omega + \frac{i\gamma}{2}\right|^2}$$

$$= \frac{1}{(\omega_0 - \omega)^2 + \frac{\gamma^2}{4}}$$

Half of maximum when

$$(\omega_0 - \omega)^2 = \frac{\gamma^2}{4}$$

$$|\omega_0 - \omega| = \frac{\gamma}{2}$$

$$\omega = \omega_0 \pm \frac{\gamma}{2}$$

So, $\boxed{\text{FWHM} = \gamma = \frac{\omega_0}{Q}}$

3. Have complex index of refraction

$$\tilde{n} = \sqrt{1+\chi} = \sqrt{1+\chi' - i\chi''}$$

for $\chi'' \ll 1$,

$$\tilde{n} \simeq \sqrt{1+\chi'}\left(1 - \frac{i\chi''}{1+\chi'}\right)^{1/2}$$

$$\simeq \sqrt{1+\chi'}\left(1 - \frac{1}{2}\frac{i\chi''}{1+\chi'}\right)$$

$$= \sqrt{1+\chi'} - \frac{1}{2}\frac{i\chi''}{\sqrt{1+\chi'}}$$

So, real part of index $n = \sqrt{1+\chi'}$

Know from optics that field propagates as

$$E = E_0 e^{-ikz}$$

where $k = \tilde{n}\frac{\omega}{c} \equiv \tilde{n} k_0$

So,

$$E = E_0 e^{-i\left(n - \frac{i\chi''}{2n}\right)k_0 z}$$

$$= E_0 e^{-\frac{\chi''}{2n}k_0 z} e^{-ink_0 z}$$

Then $I \propto |E|^2$; so.

$$I = I_0 e^{-\frac{\chi''}{n}k_0 z}$$

and $\alpha = \frac{\chi''}{n}k_0 = \boxed{\frac{\chi''\omega}{nc}} = \frac{\chi''\omega}{\sqrt{1+\chi'}\,c}$

4. (a) Can assume $\gamma_R \ll \omega_0$, so take position of electron as

$$x(t) = x_0 \cos \omega_0 t$$

to first approximation

Then
$$v(t) = -\omega x_0 \sin \omega_0 t$$
$$\dot{v}(t) = -\omega^2 x_0 \cos \omega_0 t$$

so
$$P_{rad} = \frac{e^2}{6\pi\varepsilon_0 c^3} \omega_0^4 x_0^2 \cos^2 \omega_0 t$$

time average
$$P_{rad} = \frac{e^2 \omega_0^4 x_0^2}{12\pi\varepsilon_0 c^3}$$

But energy of electron
$$E = \frac{1}{2} m v^2 + \frac{1}{2} m \omega^2 x^2$$
$$= \frac{1}{2} m \omega_0^2 x_0^2 (\sin^2 \omega_0 t + \cos^2 \omega_0 t)$$
$$= \frac{1}{2} m \omega_0^2 x_0^2$$

So,
$$P_{rad} = \frac{dE}{dt} = -\frac{e^2 \omega_0^2}{6\pi\varepsilon_0 m c^3} \left(\frac{1}{2} m \omega_0^2 x_0^2\right)$$
$$= -\gamma_R E$$

So,
$$\boxed{\gamma_R = \frac{e^2 \omega_0^2}{6\pi\varepsilon_0 m c^3}}$$

4, (cont)

b) With no damping, x satisfies SHO equation

$$\ddot{x} + \omega_0^2 x = 0$$

Add a damping term,

$$\ddot{x} + \xi \dot{x} + \omega_0^2 x = 0$$

gives solution $x = x_0 e^{-\xi t/2} \cos \omega_0 t$

But energy $E \propto x^2 \propto e^{-\gamma_R t}$

So, have $\xi = \gamma_R$

Also have applied field E

gives force $-eE = m\ddot{x}$

so, get equation for x:

$$\ddot{x} + \gamma_R \dot{x} + \omega_0^2 x = -\frac{e}{m} E$$

But $p = -ex$, so have

$$\boxed{\ddot{p} + \gamma_R \dot{p} + \omega_0^2 p = \frac{e^2}{m} E}$$

4. (cont)

c) A single atom radiates power $P_1 = \frac{e^2 \omega_0^4 x_0^2}{12\pi \varepsilon_0 c^3} = \frac{p_0^2 \omega_0^4}{12\pi \varepsilon_0 c^3}$

from (a)

Macroscopic polarization $\underline{P} = \frac{N}{V} p_0$

so $p_0 = \frac{V \underline{P}}{N}$

So total radiated power is

$$P_{rad} = N P_1 = N \left(\frac{\frac{V^2 \underline{P}^2}{N^2} \omega_0^4}{12\pi \varepsilon_0 c^3} \right)$$

$$\boxed{P_{rad} = \frac{V^2 \underline{P}^2 \omega_0^4}{12\pi \varepsilon_0 c^3 N}}$$

In absence of driving field,

$$\underline{P}(t) = \frac{N}{V} p(t)$$

$$\boxed{\underline{P}(t) = \frac{N}{V} p(0) e^{-\gamma_R t/2} \cos \omega_0 t}$$

d) γ_1 = energy loss, just like γ_R

So, γ_1 just adds to γ_R

For γ_2, every time collision happens, atom's dipole is dephased. Fields radiated by these atoms will add with random phases, and tend to cancel out.

So, expect than only atoms which have $\underline{\text{not}}$ collided will contribute to P_{rad}.

Number of atoms that have not collided is

$$N(t) = N(0)\, e^{-\gamma_2 t}$$

So, get $P(t) = \dfrac{N(t)}{V}\, p(t)$

microscopic dipole from unperturbed atoms

$$\propto e^{-\gamma_2 t}$$

So, expect

$$\boxed{P(t) = \frac{N}{V}\, p(0)\, e^{-\left(\frac{\gamma_R}{2} + \frac{\gamma_1}{2} + \gamma_2\right)t} \cos\omega_0 t}$$

4. (cont)

e) Above $\underline{P}(t)$ indicates differential equation

$$\ddot{\underline{P}} + (\gamma_R + \gamma_1 + 2\gamma_2)\dot{\underline{P}} + \omega_0^2 \underline{P} = 0$$

driving term is $\frac{N}{V} \times \ddot{p}$

$$= \frac{Ne^2}{mV} E(t)$$

So

$$\boxed{\ddot{\underline{P}} + (\gamma_R + \gamma_1 + 2\gamma_2)\dot{\underline{P}} + \omega_0^2 \underline{P} = \frac{Ne^2}{mV} E(t)}$$

Steady state, $\underline{P} = \underline{P}_0 e^{i\omega t}$

$$-\omega^2 \underline{P}_0 + i\omega\Gamma\underline{P}_0 + \omega_0^2\underline{P}_0 = \frac{Ne^2}{mV} E_0$$

$$\Gamma \equiv \gamma_R + \gamma_1 + 2\gamma_2$$

So $$\underline{P}_0 = \frac{Ne^2}{mV} \frac{1}{\omega_0^2 - \omega^2 + i\omega\Gamma} E_0$$

$$\approx \frac{Ne^2}{2m\omega_0 V} \frac{1}{\omega_0 - \omega + i\Gamma/2} E_0$$

for $\omega \approx \omega_0$

(cf. Problem 2)

$$= \frac{Ne^2}{2m\omega_0 V} \frac{(\omega_0 - \omega) - i\frac{\Gamma}{2}}{(\omega_0-\omega)^2 + \Gamma^2/4} E_0$$

4. (cont)

But

$$\underline{P}_0 = \varepsilon_0 \chi \, \underline{E}_0 = \varepsilon_0 (\chi' - i\chi'') E_0 .$$

So

$$\boxed{\begin{aligned} \chi' &= \frac{Ne^2}{2\varepsilon_0 m \omega_0 V} \frac{\omega_0 - \omega}{(\omega_0-\omega)^2 + \frac{\Gamma^2}{4}} \\ \chi'' &= \frac{Ne^2}{2\varepsilon_0 m \omega_0 V} \frac{\Gamma/2}{(\omega_0-\omega)^2 + \Gamma^2/4} \end{aligned}}$$

with $\Gamma = \gamma_R + \gamma_1 + 2\gamma_2$

5.

(a) $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

p_i = probabilities, so

$0 \leq p_i \leq 1$

if we diagonlize ρ, then we express it in exactly this form, but with $|\psi_i\rangle$'s orthogonal.

So, eigenvalues of ρ <u>are</u> $\{p_i\}$'s, so they are between 0 and 1

5. (cont)

Can always write $\rho = S^{-1} D S$ for diagonal D

So $\rho^2 = S^{-1} D^2 S$

and $\rho^2 = \rho$ if and only if $D^2 = D$

But $D^2 = D$ only if eigenvalues of ρ are zero or 1

Need $\sum \lambda_i = 1$ since $\lambda_i = p_i$, so can have only one $\lambda_i = 1$, rest are zero.

Then $\rho = |\psi_i\rangle\langle\psi_i|$ for that i, which represents a pure state.

c) Here

$$\rho = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\boxed{\rho = \frac{1}{4}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}}$$

5. (cont)

Find eigenvalues:

$$\left| \begin{bmatrix} 3/4-\lambda & 1/4 \\ 1/4 & 1/4-\lambda \end{bmatrix} \right| = 0$$

$$\left(\frac{3}{4}-\lambda\right)\left(\frac{1}{4}-\lambda\right) - \frac{1}{16} = 0$$

$$\lambda^2 - \lambda + \frac{3}{16} - \frac{1}{16} = 0$$

$$\lambda^2 - \lambda + \frac{1}{8} = 0$$

$$\lambda = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{1}{2}}\right) = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{2}}\right)$$

$$= 0.85,\ 0.15$$

So $0 \leq \lambda \leq 1$

also,

$$\rho^2 = \frac{1}{16}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \neq \rho$$

not a pure state!