Phys 532 HW 4 Solutions

1. a) $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$ $H = \hbar \begin{bmatrix} 0 & -2\Omega\cos\omega t \\ -2\Omega\cos\omega t & \omega_0 \end{bmatrix}$

$\equiv \begin{bmatrix} 0 & b \\ b & a \end{bmatrix}$ for convenience

then $[\rho, H] = \rho H - H\rho$

$$= \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 0 & b \\ b & a \end{bmatrix} - \begin{bmatrix} 0 & b \\ b & a \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b\rho_{12} & b\rho_{11} + a\rho_{12} \\ b\rho_{22} & b\rho_{21} + a\rho_{22} \end{bmatrix} - \begin{bmatrix} b\rho_{21} & b\rho_{22} \\ b\rho_{11} + a\rho_{21} & b\rho_{12} + a\rho_{22} \end{bmatrix}$$

$$= \begin{bmatrix} b(\rho_{12} - \rho_{21}) & b(\rho_{11} - \rho_{22}) + a\rho_{12} \\ b(\rho_{22} - \rho_{11}) - a\rho_{21} & b(\rho_{21} - \rho_{12}) \end{bmatrix}$$

So,

$$\left.\frac{d\rho}{dt}\right|_{coh} = \begin{bmatrix} -2i\Omega\cos\omega t\,(\rho_{12} - \rho_{21}) & -2i\Omega\cos\omega t\,(\rho_{11} - \rho_{22}) + i\omega_0\rho_{12} \\ -2i\Omega\cos\omega t\,(\rho_{22} - \rho_{11}) - i\omega_0\rho_{21} & -2i\Omega\cos\omega t\,(\rho_{21} - \rho_{12}) \end{bmatrix}$$

b) Solve $\left.\frac{d\rho}{dt}\right|_{inc} = 0 = \begin{bmatrix} -\Gamma_1\rho_{11} + \Gamma_2\rho_{22} & -\gamma\rho_{12} \\ -\gamma\rho_{21} & -\Gamma_2\rho_{22} + R \end{bmatrix}$

So $\rho_{12} = \rho_{21} = 0$.

$\Gamma_1\rho_{11} = \Gamma_2\rho_{22}$

$\Gamma_2\rho_{22} = R \quad \Rightarrow \quad \rho_{22} = \frac{R}{\Gamma_2}$

so $\rho_{11} = \frac{R}{\Gamma_1}$

Then $\boxed{\Delta N_0 = NR\left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2}\right)}$

c) We have:

$$\frac{d\rho_{11}}{dt} = 2i\Omega\cos\omega t(\rho_{21}-\rho_{12}) - \Gamma_1\rho_{11} + \Gamma_2\rho_{22}$$

$$\frac{d\rho_{22}}{dt} = 2i\Omega\cos\omega t(\rho_{12}-\rho_{21}) - \Gamma_2\rho_{22} + R$$

$$\frac{d\rho_{12}}{dt} = 2i\Omega\cos\omega t(\rho_{22}-\rho_{11}) + (i\omega_0-\gamma)\rho_{12}$$

Rotating wave approximation:

define, $\sigma_{12} = \rho_{12}e^{-i\omega t}$

$$\frac{d\sigma_{12}}{dt} = \frac{d\rho_{12}}{dt} e^{-i\omega t} - i\omega \sigma_{12}$$

$$= 2i\Omega e^{-i\omega t} \cos\omega t (\rho_{22} - \rho_{11}) + (i\omega_0 - i\omega - \gamma)\sigma_{12}$$

approximate $e^{-i\omega t} \cos\omega t \to \frac{1}{2}$

$$\frac{d\sigma_{12}}{dt} = -(i\Delta + \gamma)\sigma_{12} + i\Omega(\rho_{22} - \rho_{11})$$

$\Delta = \omega - \omega_0$

also:

$$\frac{d\rho_{11}}{dt} = 2i\Omega \cos\omega t (e^{-i\omega t}\sigma_{21} - e^{i\omega t}\sigma_{12}) - \Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}$$

$$\simeq i\Omega(\sigma_{21} - \sigma_{12}) - \Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}$$ in approximation

and

$$\frac{d\rho_{22}}{dt} = i\Omega(\sigma_{12} - \sigma_{21}) - \Gamma_2 \rho_{22} + R$$

Set derivatives to 0:

$$0 = -(i\Delta + \gamma)\sigma_{12} + i\Omega(\rho_{22} - \rho_{11}) \quad (1)$$

$$0 = i\Omega(\sigma_{21} - \sigma_{12}) - \Gamma_1 \rho_{11} + \Gamma_2 \rho_{22} \quad (2)$$

$$0 = i\Omega(\sigma_{12} - \sigma_{21}) - \Gamma_2 \rho_{22} + R \quad (3)$$

So, $$\sigma_{12} = \frac{\Delta N_0}{N} \frac{\Gamma_1 \Gamma_2}{\Gamma_2 - \Gamma_1} \frac{1}{\Gamma_1} \frac{\Gamma_1 - \Gamma_2}{2\Omega^2\gamma + \Delta^2\Gamma_2 + \gamma^2\Gamma_2} \Omega(\Delta + i\gamma)$$

$$= - \frac{\Delta N_0}{N} \frac{\Gamma_2}{2\Omega^2\gamma + \Delta^2\Gamma_2 + \gamma^2\Gamma_2} \cdot \Omega(\Delta + i\gamma)$$

$$\boxed{\sigma_{12} = \frac{\Delta N_0}{N} \frac{-\Omega(\Delta + i\gamma)}{\Delta^2 + \gamma^2 + 2\Omega^2 \frac{\gamma}{\Gamma_2}}}$$

d) $$\langle p \rangle = \mu(\rho_{12} + \rho_{21})$$

$$= \mu\left(\sigma_{12} e^{i\omega t} + \sigma_{21} e^{-i\omega t}\right)$$

$$= \mu\left[(\sigma_{12} + \sigma_{21})\cos\omega t + i(\sigma_{12} - \sigma_{21})\sin\omega t\right]$$

$$= \mu \frac{\Delta N_0}{N} \frac{(-\Omega)}{\Delta^2 + \gamma^2 + 2\Omega^2 \frac{\gamma}{\Gamma_2}} \left[2\Delta\cos\omega t - 2\gamma\sin\omega t\right]$$

Use $\Omega = \frac{\mu E_0}{2\hbar}$, and $P = N\langle p \rangle$

$$P = \Delta N_0 \frac{\mu^2 E_0}{\hbar} \frac{1}{\Delta^2 + \gamma^2 + 2\Omega^2 \frac{\gamma}{\Gamma_2}} (-\Delta\cos\omega t + \gamma\sin\omega t)$$

Then

$$-A\left(\rho_{22} - \frac{R}{\Gamma_1}\right) - \Gamma_2 \rho_{22} + R = 0$$

$$\rho_{22}(A+\Gamma_2) = R + \frac{AR}{\Gamma_1}$$

$$\rho_{22} = R\,\frac{1+\frac{A}{\Gamma_1}}{A+\Gamma_2} = \boxed{\frac{R}{\Gamma_1}\,\frac{A+\Gamma_1}{A+\Gamma_2}}$$

So
$$\sigma_{12} = \frac{\Omega(\Delta+i\gamma)}{\Delta^2+\gamma^2}\,\frac{-R}{\Gamma_1}\left(\frac{A+\Gamma_1}{A+\Gamma_2} - 1\right)$$

$$= \frac{\Omega(\Delta+i\gamma)}{\Delta^2+\gamma^2}\,\frac{R}{\Gamma_1}\,\frac{\Gamma_1-\Gamma_2}{A+\Gamma_2}$$

$$= \frac{\Omega(\Delta+i\gamma)}{\Delta^2+\gamma^2}\,\frac{R}{\Gamma_1}\,\frac{\Gamma_1-\Gamma_2}{\frac{2\Omega^2\gamma}{\Delta^2+\gamma^2}+\Gamma_2}$$

$$= \Omega(\Delta+i\gamma)\,\frac{R}{\Gamma_1}\,\frac{\Gamma_1-\Gamma_2}{2\Omega^2\gamma+\Gamma_2\Delta^2+\Gamma_2\gamma^2}$$

Express in terms of $\Delta N_0 = NR\left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2}\right)$

$$R = \frac{\Delta N_0}{N}\,\frac{1}{\frac{1}{\Gamma_1}-\frac{1}{\Gamma_2}}$$

$$= \frac{\Delta N_0}{N}\,\frac{\Gamma_1\Gamma_2}{\Gamma_2-\Gamma_1}$$

From (1),

$$\sigma_{12} = \frac{i\Omega}{i\Delta+\gamma}(\rho_{22}-\rho_{11})$$

$$= \frac{i\Omega(-i\Delta+\gamma)}{\Delta^2+\gamma^2}(\rho_{22}-\rho_{11})$$

$$= \frac{\Omega(\Delta+i\gamma)}{\Delta^2+\gamma^2}(\rho_{22}-\rho_{11})$$

and $\sigma_{21} = \sigma_{12}^* = \dfrac{\Omega(\Delta-i\gamma)}{\Delta^2+\gamma^2}(\rho_{22}-\rho_{11})$

So $\sigma_{21} - \sigma_{12} = \dfrac{-2i\gamma\Omega}{\Delta^2+\gamma^2}(\rho_{22}-\rho_{11})$

Define $A = \dfrac{2\gamma\Omega^2}{\Delta^2+\gamma^2}$. Then (2) & (3) become

$$A(\rho_{22}-\rho_{11}) - \Gamma_1\rho_{11} + \Gamma_2\rho_{22} = 0$$

$$-A(\rho_{22}-\rho_{11}) - \Gamma_2\rho_{22} + R = 0$$

add: $-\Gamma_1\rho_{11} + R = 0$

$$\boxed{\rho_{11} = \frac{R}{\Gamma_1}}$$

Have

$$P = Re\left\{ \varepsilon_0 (\chi' - i\chi'') E_0 e^{i\omega t} \right\}$$

$$= \varepsilon_0 \chi' E_0 \cos\omega t + \varepsilon_0 \chi'' E_0 \sin\omega t$$

So

$$\chi' = -\Delta N_0 \frac{\mu^2}{\varepsilon_0 \hbar} \frac{\Delta}{\Delta^2 + \gamma^2 + 2\Omega^2 \frac{\gamma}{\Gamma_2}}$$

$$\chi'' = \Delta N_0 \frac{\mu^2}{\varepsilon_0 \hbar} \frac{\gamma}{\Delta^2 + \gamma^2 + 2\Omega^2 \frac{\gamma}{\Gamma_2}}$$

Write $\gamma = \frac{1}{T_2}$ $\Gamma_2 = \frac{2}{\tau}$

$$\boxed{\begin{aligned} \chi' &= -\Delta N_0 \frac{\mu^2}{\varepsilon_0 \hbar} \frac{\Delta T_2^2}{1 + \Delta^2 T_2^2 + 4 T_2 \tau \Omega^2} \\ \chi'' &= \Delta N_0 \frac{\mu^2}{\varepsilon_0 \hbar} \frac{T_2}{1 + \Delta^2 T_2^2 + 4 T_2 \tau \Omega^2} \end{aligned}}$$

as desired

2. Have

$$\frac{t_c}{t_q} = \frac{6\pi \varepsilon_0 m c^3}{\omega_0^2 e^2} \cdot \frac{\pi 2^{19} \hbar^3 \omega_0^3 \varepsilon_0}{3^{11} m^2 c^3 e^2}$$

$$= \pi^2 \frac{2^{20}}{3^{10}} \frac{\hbar^3 \omega_0 \varepsilon_0^2}{m e^4}$$

From Yariv section 8.3, have

$$\omega_0 = \frac{3}{4} \frac{m e^4}{32\pi^2 \varepsilon_0^2 \hbar^3}$$

so

$$\frac{t_c}{t_q} = \pi^2 \frac{2^{20}}{3^{10}} \frac{\hbar^3 \varepsilon_0^2}{m e^4} \cdot \frac{3}{2^7} \frac{m e^4}{\varepsilon_0^2 \hbar^3}$$

$$= \boxed{\frac{2^{13}}{3^9}}$$

3. Absorption coefficient

$$\alpha = \frac{2\pi}{\lambda} \chi''$$

with $\chi'' = \frac{\pi c^3}{2\omega^3 t_s} \Delta N_0 \, g(\nu)$

where $\Delta N_0 = N_1 \frac{g_2}{g_1} - N_2$

$$\simeq N \frac{g_2}{g_1}$$

Since $N_2 \approx 0$ in absence of puming

3. Broadening mechanisms:

Radiative: $\Delta\nu = \frac{1}{\pi T_2}$

for radiation only, $T_2 = 2t_s$

$$\Delta\nu_R = \frac{1}{2\pi t_s} = 10 \text{ MHz}$$

Collisions: $\Delta\nu = \frac{1}{\pi T_2}$

for collisions only, $T_2 = T_{collision} = 1 \text{ ms}$

$$\Delta\nu = \frac{1}{\pi \times 1\text{ms}} = 320 \text{ Hz}$$

Doppler:

$$\Delta\nu_D = \left(\frac{8 k_B T}{M\lambda^2} \ln 2\right)^{1/2}$$

$$= 1{,}5 \text{ GHz}$$

for $T = 400\text{K}$

$M = 23 \times 1.67 \times 10^{-27} \text{ kg}$

So, Doppler broadening is dominant

Then $\alpha = \frac{1}{8\pi} \frac{\lambda^2}{t_s} N \frac{g_2}{g_1} \left(\frac{M\lambda^2}{2\pi kT}\right)^{1/2}$

$$= \frac{1}{8\pi} \frac{(589\text{nm})^2}{16\text{ns}} 3\times10^{17}\text{m}^{-3} \left(6.2\times10^{-10}\text{s}\right) \times \frac{g_2}{g_1}$$

$$= 160 \text{ m}^{-1} \frac{g_2}{g_1} = 1.6 \text{ cm}^{-1} \frac{g_2}{g_1}$$

So for $3p_{3/2}$ state, $\boxed{\alpha = 3.2 \text{ cm}^{-1}}$

for $3p_{1/2}$, $\boxed{\alpha = 1.6 \text{ cm}^{-1}}$

4. Saturation intensity is defined as

$$I_s = \frac{4\pi h\nu}{\lambda^2} \frac{t_s}{\tau} \frac{1}{g(\nu_0)}$$

$$= \frac{2\pi hc}{\lambda^3 T_2} \quad ; \quad \text{since } \tau \approx t_s, \quad g(\nu_0) = 2T_2$$

But for radiative broadening, $T_2 = 2t_s$

$$I_s = \frac{\pi hc}{\lambda^3 t_s} = \boxed{190 \ \frac{W}{m^2}}$$

This is the same for both transitions, since the degeracy doesn't matter.

5. In general

$$g(\nu) = \int_{-\infty}^{\infty} d\nu' \, p(\nu') \, g_H(\nu, \nu')$$

where $p(\nu') =$ prob that atom has resonance at $\nu = \nu'$

and
$$g_H(\nu, \nu') = \frac{2T_2}{1 + 4\pi^2 T_2^2 (\nu - \nu')^2}$$

$$= \frac{4t_s}{1 + 16\pi^2 t_s^2 (\nu - \nu')^2}$$

for $T_2 = 2t_s$

What is $p(\nu)$?

Know $p(z) =$ prob that atom is at position z

$$p(z) = \begin{cases} \frac{1}{L} & \text{for } -\frac{L}{2} < z < \frac{L}{2} \\ 0 & \text{for } |z| > \frac{L}{2} \end{cases}$$

and $p(\nu) = \frac{dz}{d\nu} p(z)$

where $\nu(z) = \nu_0 + \frac{h B(z)}{2\pi}$

$$= \nu_0 + \frac{h \beta z}{2\pi}$$

So $\frac{dz}{d\nu} = \left(\frac{d\nu}{dz}\right)^{-1} = \left(\frac{h\beta}{2\pi}\right)^{-1} = \frac{2\pi}{h\beta}$

5. (cont)

So,

$$\rho(\nu) = \begin{cases} \dfrac{2\pi}{h\beta L} & \text{for} \quad \nu_0 - \dfrac{h\beta L}{4\pi} < \nu < \nu_0 + \dfrac{h\beta L}{4\pi} \\ 0 & \text{for} \quad |\nu - \nu_0| > \dfrac{h\beta L}{4\pi} \end{cases}$$

So,

$$g(\nu) = \int_{\nu_0 - \frac{h\beta L}{4\pi}}^{\nu_0 + \frac{h\beta L}{4\pi}} \frac{2\pi}{h\beta L} \frac{4t_s}{1 + 16\pi^2 t_s^2 (\nu - \nu')^2} \, d\nu'$$

(a) If $\frac{1}{t_s} \ll h\beta L$, then as long as $|\nu - \nu_0| < \frac{h\beta L}{4\pi}$, limits can be taken to be $\pm\infty$, so

$$\boxed{\begin{aligned} g(\nu) &= \frac{2\pi}{h\beta L} & |\nu - \nu_0| < \frac{h\beta L}{4\pi} \\ &= 0 & |\nu - \nu_0| > \frac{h\beta L}{4\pi} \end{aligned}}$$

(Since we know $\int_{-\infty}^{\infty} d\nu \, g_H(\nu) = 1$)

(b) In general, need to do integral

$$g(\nu) = \frac{8\pi t_s}{h\beta L} \int_{\nu_0 - \delta}^{\nu_0 + \delta} \frac{1}{1 + 16\pi^2 t_s^2 (\nu - \nu')^2} \, d\nu'$$

where $\delta = \frac{h\beta L}{4\pi}$

5. (cont)

Define $u = 4\pi t_s(\nu' - \nu)$ so $\nu' = \nu + \frac{u}{4\pi t_s}$

Then

$$g(\nu) = \frac{8\pi t_s}{h\beta L} \int_{4\pi t_s(\nu_0-\nu-\delta)}^{4\pi t_s(\nu_0-\nu+\delta)} \frac{\frac{du}{4\pi t_s}}{1+u^2}$$

$$= \frac{2}{h\beta L} \int_{a-b}^{a+b} \frac{du}{1+u^2}$$

$a = 4\pi t_s(\nu_0 - \nu)$

$b = 4\pi t_s \delta = t_s h\beta L$

$$g(\nu) = \frac{2}{h\beta L}\left[\tan^{-1}(a+b) - \tan^{-1}(a-b)\right]$$

Simplify:

$$\tan\left[\frac{h\beta L}{2} g(\nu)\right] = \tan\left[\tan^{-1}(a+b) - \tan^{-1}(a-b)\right]$$

but $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

So, $$\tan\left[\frac{h\beta L}{2} g(\nu)\right] = \frac{(a+b)-(a-b)}{1+(a+b)(a-b)} = \frac{2b}{1+a^2-b^2}$$

5. (cont)

Thus

$$g(\nu) = \frac{2}{h\beta L} \tan^{-1} \frac{2b}{1+a^2-b^2}$$

$$\boxed{g(\nu) = \frac{2}{h\beta L} \tan^{-1} \frac{2t_s h\beta L}{1+16\pi^2 t_s^2(\nu-\nu_0)^2 - t_s^2 h^2\beta^2 L^2}}$$

Sketch: if $t_s h\beta L = 1$,

$$g(\nu) = 2t_s \tan^{-1} \frac{1}{8\pi^2 t_s^2(\nu-\nu_0)^2}$$

at $\nu = \nu_0$, $\tan^{-1}\infty = \frac{\pi}{2}$

$$g(\nu_0) = \pi t_s$$

Falls to $\frac{1}{2}$ max at

$$\tan^{-1} \frac{1}{8\pi^2 t_s^2(\nu-\nu_0)^2} = \frac{\pi}{4}$$

$$\frac{1}{8\pi^2 t_s^2(\nu-\nu_0)^2} = 1$$

$$|\nu-\nu_0| = \frac{1}{\sqrt{8}\,\pi t_s}$$