Phys 532 Assigment 8 Solutions

1. a) Polarization entering crystal is

$$\hat{\varepsilon}_{in} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}) \quad \text{for } \hat{y} \text{ vertical}, \ \hat{x} \text{ horizontal}$$

Polarization coming out is

$$\hat{\varepsilon}_{out} = \frac{1}{\sqrt{2}}(\hat{x} + e^{i\phi}\hat{y})$$

Then fraction transmitted through polarizer is

$$T = |\hat{\varepsilon}_{out} \cdot \hat{t}|^2 \quad \text{for } \hat{t} = \text{transmission axis of polarizer}$$

$$= \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$$

$$T = \left|\tfrac{1}{2}(1 - e^{i\phi})\right|^2 = \tfrac{1}{4}\left[(1-\cos\phi)^2 + \sin^2\phi\right]$$

$$= \tfrac{1}{4}\left[1 - 2\cos\phi + \cos^2\phi + \sin^2\phi\right]$$

$$= \tfrac{1}{2}\left[1 - \cos\phi\right]$$

But $\sin^2\frac{\phi}{2} = \frac{1}{2}(1-\cos\phi)$

So $T = \sin^2\frac{\phi}{2}$

and $\boxed{I_{out} = I_{in}\sin^2\frac{\phi}{2}}$

1. (b)

With $\lambda/4$ plate, y component picks up an addition $\frac{\pi}{2}$ phase shift

So $\phi \to k l \Delta n + \frac{\pi}{2}$

Still have $T = \sin^2 \frac{\phi}{2}$

but now $T = \sin^2 \frac{k l \Delta n + \frac{\pi}{2}}{2}$

$= \sin^2\left(\frac{k l \Delta n}{2} + \frac{\pi}{4}\right)$

For small Δn, Taylor expand $\sin^2\theta$

$\sin^2\left(\frac{\pi}{4} + \varepsilon\right) = \sin^2\frac{\pi}{4} + 2\sin\frac{\pi}{4}\cos\frac{\pi}{4}\,\varepsilon$

$= \frac{1}{2} + \varepsilon$

So $T = \frac{1}{2} + \frac{k l \Delta n}{2}$

$$\boxed{I_{out} = \frac{1}{2} I_{in}(1 + k l \Delta n)}$$

linear response as desired

2. Since $\hat{k}$ is in x-z plane, light can be polarized along y, so $\hat{y}$ will be on principal polarization with index $\boxed{n_{y'} = n_y}$

Other direction is in x-z plane, but perpedicular to $\hat{k}$.

So $\hat{x}' = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}$

To find index $n_{x'}$, use ellipsoid

$$\frac{1}{n_{x'}^2} = \frac{x^2}{n_x^2} + \frac{z^2}{n_z^2} \quad \text{for} \quad x = \cos\theta, \quad z = -\sin\theta$$

$$\boxed{\frac{1}{n_{x'}^2} = \frac{\cos^2\theta}{n_x^2} + \frac{\sin^2\theta}{n_z^2}}$$

3. From Table 14.1, $LiNbO_3$ is type 3m.
Take standard orientation, then have

$r_{13} = r_{23} = 8.6$

$-r_{12} = +r_{22} = -r_{61} = 3.4$

$r_{33} = 30.8$

$r_{42} = r_{51} = 28$

Values from table 14.2, in $10^{-12}\ \frac{m}{V}$

Also
$n_1 = n_2 = n_o = 2.29$
$n_e = 2.20$

3 (a) Index ellipsoid

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} + r_{13} x^2 E_z + r_{13} y^2 E_z - r_{22} x^2 E_y$$

$$+ r_{22} y^2 E_y - 2 r_{22} x y E_z$$

$$+ r_{33} z^2 E_z + 2 r_{42} y z E_y + 2 r_{42} x z E_x = 1$$

or

$$\boxed{\left(\frac{1}{n_o^2} + r_{13} E_z - r_{22} E_y\right) x^2 + \left(\frac{1}{n_o^2} + r_{13} E_z + r_{22} E_y\right) y^2 + \left(\frac{1}{n_e^2} + r_{33} E_z\right) z^2 + 2 r_{42} y z E_y + 2 r_{42} x z E_x - 2 r_{22} x y E_x = 1}$$

b) For modulator, would like polarization in x-y plane to avoid thermal drifts

So, take beam propagating along $\hat{z}$.

Then index ellipse is

$$\left(\frac{1}{n_o^2} + r_{13} E_z - r_{22} E_y\right) x^2 + \left(\frac{1}{n_o^2} + r_{13} E_z + r_{22} E_y\right) y^2$$

$$- 2 r_{22} x y E_x = 1$$

So, could apply field along x, y, or z.

E_x & E_y have advantage that field is $\perp$ to transmission axis, so get larger field for given voltage.

But consider all 3:

E_x: have $\frac{1}{n_o^2} x^2 + \frac{1}{n_o^2} y^2 - 2 r_{22} xy E_x = 1$

Like KDP. Get new principal indices

$$\frac{1}{n_{x'}^2} = \frac{1}{n_o^2} + r_{22} E_x$$

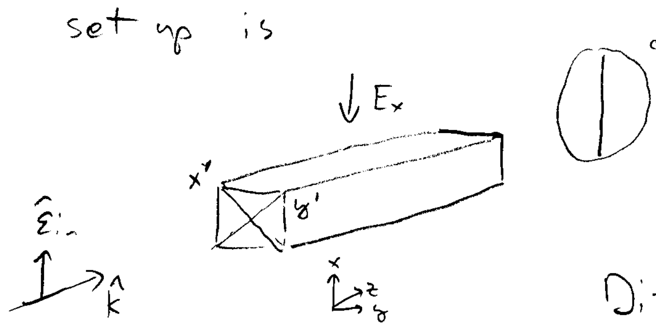
or $\Delta n_{x'} = \frac{1}{2} n_o^3 r_{22} E_x$

and $\frac{1}{n_{y'}^2} = \frac{1}{n_o^2} - r_{22} E_y$

$$\Delta n_{y'} = -\frac{1}{2} n_o^3 r_{22} E_x$$

with x', y' at $45°$ to x, y

So set up is



$\hat{\varepsilon}_{in} = \hat{x}$ or $\hat{y}$

Differential phase

$$\boxed{\phi = k \ell n_o^3 r_{22} E_x}$$

E_y:

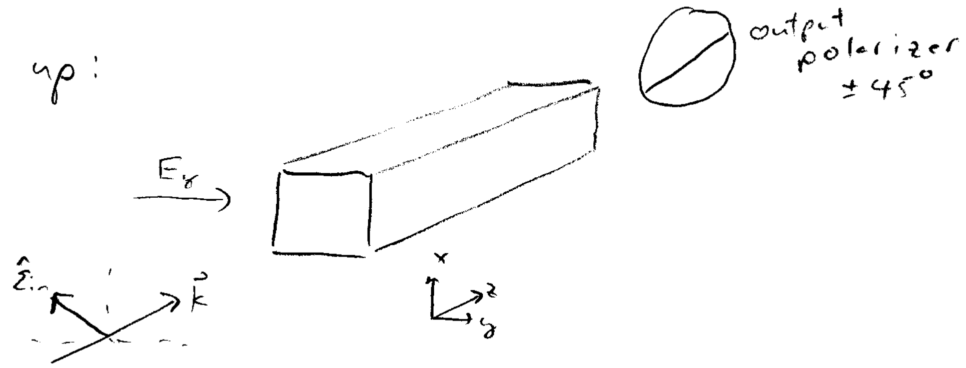
Index ellipse is

$$\left(\frac{1}{n_o^2} - r_{22}E_y\right)x^2 + \left(\frac{1}{n_o^2} + r_{22}E_y\right)y^2 = 1$$

Already diagonal, x & y axes remain good

$$\Delta n_x = \frac{1}{2} n_o^3 r_{22} E_y \qquad \Delta n_y = \frac{1}{2} n_o^3 r_{22} E_y$$

Set up:



Want $\hat{\varepsilon}_{in}$ at 45° to x & y

Phase shift $\boxed{\phi = k l n_o^3 r_{22} E_y}$

E_z: Ellipse is

$$\left(\frac{1}{n_o^2} + r_{13}E_z\right)x^2 + \left(\frac{1}{n_o^2} + r_{13}E_z\right)y^2 = 1$$

Oops! This one doesn't work after all, change in n_x and n_y is the same!

(c) For either E_x or E_y case,

$$\phi = k l n_o^3 r_{22} E_y$$

$$k = \frac{2\pi}{\lambda}$$

Take $\lambda = 633\text{nm}$

Then
$$E_y = \frac{\lambda}{2l} \frac{1}{n_o^3 r_{22}} = \frac{V_\pi}{t}$$

$t =$ crystal thickness $= 1\text{mm}$

$$V_\pi = \frac{\lambda}{2} \frac{t}{l} \frac{1}{n_o^3 r_{22}}$$

$$= \frac{633\text{nm}}{2} \cdot \frac{1\text{mm}}{20\text{mm}} \frac{1}{(2.29)^3} \frac{1}{3.4\times 10^{-12}\frac{m}{V}}$$

$$\boxed{V_\pi = 387\text{ V}}$$

4. a) In general $E_{out} = E_{in} e^{-inkl}$ for vacuum $k = \frac{2\pi}{\lambda}$

Here $n = n(t) = n_e + \Delta n(t)$

$$\Delta n(t) = \frac{1}{2} n_e^3 r_{33} E_1(t)$$

for polarization and applied field along z

So,

$$E_{out}(t, x=\ell) = E_0 e^{i\omega_0 t} e^{-in_e k\ell} e^{-i\left(\frac{k\ell}{2} n_e^3 r_{33} E_1 \cos\Omega t\right)}$$

$$\propto e^{i\left[\omega_0 t - \frac{k\ell}{2} n_e^3 r_{33} E_1 \cos\Omega t\right]}$$

Which has desired form, for

$$\boxed{\delta = -\frac{k\ell}{2} n_e^3 r_{33} E_1}$$

b) If $E_L = E_0 e^{i(\omega_0 t + \delta\cos\Omega t)}$

$$\frac{dE_L}{dt} = E_0 (i\omega_0 - i\delta\Omega \sin\Omega t) e^{i(\omega_0 t + \delta\cos\Omega t)}$$

So $\omega(t) = \omega_0 - \delta\Omega \sin\Omega t$

$$\boxed{\text{Covers range } \omega_0 - \delta\Omega \text{ to } \omega_0 + \delta\Omega}$$

c) For small δ,

$$e^{i[\omega_0 t + \delta\cos\Omega t]} \approx e^{i\omega_0 t}(1 + i\delta\cos\Omega t)$$

$$= e^{i\omega_0 t}\left[1 + \frac{i\delta}{2}\left(e^{i\Omega t} + e^{-i\Omega t}\right)\right]$$

So,

$$\boxed{E_L(t) \approx e^{i\omega_0 t} + \frac{i\delta}{2} e^{i(\omega_0+\Omega)t} + \frac{i\delta}{2} e^{i(\omega_0-\Omega)t}}$$

Waves at ω_0, $\omega_0 \pm \Omega$.

d) Given

$$E_0 e^{i(\omega_0 t + \delta\cos\Omega t)} = E_0 \sum_n A_n e^{i(\omega_0 t + n\Omega t)}$$

or $$e^{i\delta\cos\Omega t} = \sum_n A_n e^{in\Omega t}$$

Looks like Fourier series, so

$$A_m = \frac{\Omega}{2\pi} \int_0^{\frac{2\pi}{\Omega}} e^{i\delta\cos\Omega t} e^{-im\Omega t} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i[\delta\cos\phi - m\phi]} d\phi$$

Write $$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\delta\cos\phi} (\cos m\phi - i\sin m\phi)\, d\phi$$

Integral of sine part vanishes since sine is odd, rest is even

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i\delta\cos\phi} \cos m\phi\, d\phi = \frac{1}{\pi} \int_0^{\pi} e^{i\delta\cos\phi} \cos m\phi\, d\phi$$

Find this integral in Abramowitz & Stegun,
pg 360

$$\frac{1}{\pi}\int_0^{\pi} e^{iz\cos\theta}\cos n\theta \, d\theta = i^n J_n(z)$$

So, $$\boxed{A_m(\delta) = i^n J_{|m|}(\delta)}$$

To get $|A_1| = |A_0|$, need $\delta \approx 1.4$, from tables.

(Note, $J_1(\delta) \to \frac{\delta}{2}$ for small δ)