1. First, solve for mode:

Round trip matrix, starting at flat mirror is

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ \frac{2}{R} & 1 + \frac{2d}{R} \end{bmatrix} = \begin{bmatrix} 1 + \frac{2d}{R} & 2d + \frac{2d^2}{R} \\ \frac{2}{R} & 1 + \frac{2d}{R} \end{bmatrix}$$

So $q = \frac{1}{C}\left[ \frac{A-D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} \right]$

$$= \pm \frac{R}{2} \sqrt{\left(1 + \frac{2d}{R}\right)^2 - 1} = \pm \frac{R}{2} \sqrt{\frac{4d}{R} + \frac{4d^2}{R^2}}$$

$$= \pm 50\text{ cm} \sqrt{-2 + 1}$$

$$= i\,50\text{ cm}$$

So focus at back mirror, $z_0 = 50\text{cm}$

$$w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = 0.89\text{ mm}$$

Width at front mirror

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_0^2}} = w_0 \sqrt{1 + 1} = \sqrt{2}\, w_0$$

$$= 1.26\text{ mm}$$

So, estimate $\boxed{w = 1\text{ mm}}$ in cavity

Second, get medium characteristics.

density: $P = N k_B T$

$P = 1 \text{ atm} = 101325 \ N/m^2$

$T = 300 K$

So $N_a = \dfrac{101325 \frac{N}{m^2}}{1.38 \times 10^{-23} J/K \cdot 300 K} = \boxed{2.45 \times 10^{25} \ m^{-3}}$

linewidth:

Radiation $\Delta\nu = \dfrac{1}{2\pi t_1} = 160 \ kHz$

Collisions $\Delta\nu = \dfrac{1}{\pi} f_{col} = 320 \ MHz$

Doppler $\Delta\nu = 2.35 \sqrt{\dfrac{kT}{M\lambda^2}} = 52 \ MHz$

$M = 200 \times 1.66 \times 10^{-27} \ kg = 3.32 \times 10^{-25} \ kg$

So collisional broadening dominates

Third, find inversion:

For ideal four-level system with no degeneracies, $\Delta N = R(\tau_2 - \tau_1) \approx R\tau_2$

and $R = W_p N_0$

Since $W_p$ is very small, expect that we can ignore depletion of ground state

So $\Delta N = W_p \tau_2 N_a = (10^{-6} s^{-1})(10^{-3} s)(2.45 \times 10^{25} m^{-3})$

$= 2.45 \times 10^{16} \ m^{-3}$

So small signal gain is

$$\gamma_0 = \frac{\lambda^2}{8\pi t_s} g(\nu) \Delta N$$

Assume $\nu \approx \nu_0$, so

$$g(\nu) \to \frac{2}{\pi \Delta\nu} = \frac{2}{f_{col}}$$

$$\gamma_0 = \frac{\lambda^2}{4\pi^2 t_s \Delta\nu} \Delta N = \frac{\lambda^2 \tau_2 W_p N_a}{4\pi t_s f_{col}} = \frac{\lambda^2 W_p N_a}{4\pi f_{col}}$$

$$= \frac{(5\times10^{-6}\,m)^2}{4\cdot\pi^2\cdot(10^{-3}s)(320\,MHz)} 2.45\times10^{16}\,m^{-3}$$

$$\boxed{\gamma_0 = 0.048\,m^{-1}}$$

and $g_0 = 2\gamma_0 d = 0.048 > T$, so above threshold.

Also need $I_{sat} = \dfrac{4\pi^2 h c t_s \Delta\nu}{\lambda^3 \tau_s} = \dfrac{4\pi h c f_{col}}{\lambda^3}$

For ideal four-level system, $\tau_s = \tau_2 = t_s$

$$I_s = \frac{4\pi^2(6.63\times10^{-34}\,Js)(3\times10^8\,\tfrac{m}{s})(320\,MHz)}{(5\,\mu m)^3}$$

$$\boxed{I_s = 20\,\frac{W}{m^2}}$$

So $P_{out} \approx \pi w^2 I_s T \left(\frac{g_0}{T} - 1\right)$

$$= \pi(1\,mm)^2(20\,\tfrac{W}{m^2})(0.02)\left(\frac{0.048}{0.02} - 1\right)$$

$$\boxed{P_{out} = 1.7\,\mu W}$$

2.

Rate equations

$$\frac{dN_2}{dt} = +W_p N_0 - W_p N_2 - \frac{1}{\tau_2} N_2 = 0$$

$$\frac{dN_1}{dt} = +\frac{1}{\tau_{21}} N_2 - \frac{1}{\tau_1} N_1 = 0 \qquad \Rightarrow \quad N_1 = \frac{\tau_1}{\tau_{21}} N_2$$

$$\frac{dN_4}{dt} = +\frac{1}{\tau_{24}} N_2 - \frac{1}{\tau_4} N_4 = 0 \qquad \Rightarrow \quad N_4 = \frac{\tau_4}{\tau_{24}} N_2$$

$$N_2 = N_0 \frac{W_p}{W_p + \frac{1}{\tau_2}} = N_0 \frac{\tau_2 W_p}{1 + \tau_2 W_p}$$

In terms of $N_a$:

$$N_a = N_0 + N_1 + N_2 + N_4$$

$$= N_0 \left( 1 + \frac{\tau_2 W_p}{1 + \tau_2 W_p} \left[ 1 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right] \right)$$

$$= \frac{N_0}{1 + \tau_2 W_p} \left[ 1 + \tau_2 W_p \left( 2 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right) \right]$$

$$\Delta N = N_2 - N_1 = N_2 \left( 1 - \frac{\tau_1}{\tau_{21}} \right)$$

$$= N_0 \frac{\tau_2 W_p}{1 + \tau_2 W_p} \left( 1 - \frac{\tau_1}{\tau_{21}} \right)$$

$$\boxed{\Delta N = \frac{\tau_2 W_p (1 - \tau_1/\tau_{21})}{1 + \tau_2 W_p \left( 2 + \frac{\tau_4}{\tau_{24}} + \frac{\tau_1}{\tau_{21}} \right)} N_a}$$

3. Actually, this problem is trickier than I'd intended, because there are two fairly different solutions.

I'll go through general procedure:

Know that at first mirror, $q = iz_0 = i\dfrac{\pi w_0^2}{\lambda}$.

To get beam at second mirror, use

$$M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{d}{f} & 2d - \frac{d^2}{f} \\ \frac{2}{R} - \frac{1}{f} - \frac{2d}{Rf} & 1 + \frac{4d}{R} - \frac{d}{f} - \frac{2d^2}{Rf} \end{bmatrix}$$

So $$q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D} = \frac{Aiz_0 + B}{Ciz_0 + D} = iz_0$$

Since want $q_{out} = iz_0$

Then $$Aiz_0 + B = -Cz_0^2 + Diz_0$$

$$B + i(A-D)z_0 + Cz_0^2 = 0$$

Real & imaginary parts both zero:

$$A = D$$
$$B + Cz_0^2 = 0$$

$A = D$ implies $$1 - \frac{d}{f} = 1 - \frac{d}{f} + \frac{4df - 2d^2}{Rf}$$

So $$\frac{2df - d^2}{Rf} = 0$$

Actually two physically reasonable solutions:

i) $d = 2f$

ii) $R = \infty$

(Solution $d=0$ is not physical.)

Consider (ii) first:

Still need $B + C z_0^2 = 0$. with $R = \infty$, $B = 2d - \frac{d^2}{f}$

$C = -\frac{1}{f}$

$$2d - \frac{d^2}{f} - \frac{z_0^2}{f} = 0$$

$$\boxed{f = \frac{d^2 + z_0^2}{2d}}$$

Stability: can just consider M since cavity is symmetric

Stable if $\left|\frac{A+D}{2}\right| \leq 1$

here $A = D = 1 - \frac{d}{f}$, so

$$\left|1 - \frac{d}{f}\right| \leq 1$$

$1 - \frac{d}{f} < 1$ and $1 - \frac{d}{f} > -1$

$-\frac{d}{f} < 0$ $\qquad$ $\frac{d}{f} < 2$

$\Rightarrow f > 0$ $\qquad$ $f > \frac{d}{2}$

But we have $f = \frac{d}{2} + \frac{z_0^2}{2d} > \frac{d}{2}$, so cavity is $\boxed{\text{stable.}}$

Now consider (i);

If $f = \frac{d}{2}$, then $B = 2d - 2d = 0$

$$C = \frac{2}{R} - \frac{2}{d} - \frac{4}{R} = -2\left(\frac{1}{R} + \frac{1}{d}\right)$$

Need

$$B + C z_0^2 = 0$$

$$-2\left(\frac{1}{R} + \frac{1}{d}\right) z_0^2 = 0$$

$$\Rightarrow \boxed{R = -d} \quad \text{(independent of } z_0\text{)}$$

In this case, $A = -1$

$$D = 1 + \frac{4d}{-d} - \frac{d}{d/2} - \frac{2d^2}{(-d)(d/2)}$$

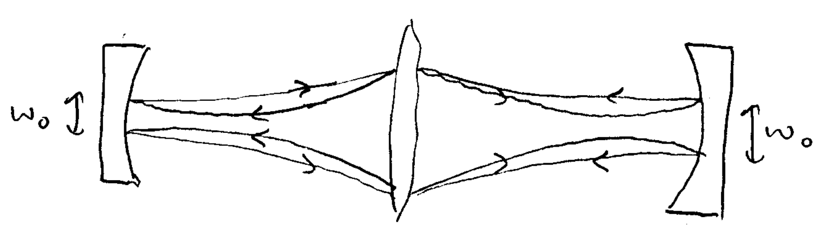
$$= 1 - 4 - 2 + 4 = -1$$

So $\frac{A+D}{2} = -1$, $\left|\frac{A+D}{2}\right| = \underline{1}$

So cavity is marginally stable: stable, but right on the border.

Thats why same cavity can support modes with different $w_0$'s.

Note, this mode is not symmetric, looks like:




But it is a valid solution.

4. Basic principle is that for lasing to occur, have gain = loss.

In one pass through medium, intensity is increased by factor $G \equiv \dfrac{I(z=l)}{I(z=0)}$

and is decreased by factor $\dfrac{I_{reflect}}{I_{incident}} = R = 1-T$

So if $I$ = intensity just before output coupler,
and $I'$ = intensity after one round trip,

then
$$I' = G \times (1-T) \times I$$

In steady-state, $I' = I$, so $G = \dfrac{1}{1-T}$

Our equation says
$$\ln \frac{I(l)}{I(0)} + \frac{I(l) - I(0)}{I_s} = \gamma_0 l$$

$$\ln G + \frac{I(l)}{I_s}\left(1 - \frac{1}{G}\right) = \gamma_0 l$$

So solve for $I(l)$ = intensity exitting medium
= intensity incident on output coupler

$$I(l) = I_s \frac{1}{1-\frac{1}{G}}\left[\gamma_0 l - \ln G\right]$$

or
$$I(l) = I_s \frac{1}{1-(1-T)}\left[\gamma_0 l - \ln \frac{1}{1-T}\right]$$

$$= I_s \frac{1}{T}\left[\gamma_0 l + \ln(1-T)\right]$$

Then output intensity is $I_{out} = T\, I(l)$

$$\boxed{I_{out} = I_s\left[\gamma_0 l + \ln(1-T)\right]}$$

Threshold when $I_{out} = 0$

or $$\boxed{\gamma_t = -\frac{1}{l} \ln(1-T)}$$

and output power $P = \pi w^2 I_{out}$

$$\boxed{P = \pi w^2 I_s \left[ \gamma_0 l - \ln(1-T) \right]}$$

5. $P_o = \pi w^2 I_s T \left( \frac{g_0}{T+L} - 1 \right)$

If $g_0 \gg T+L$, then

$$P_{out} = \pi w^2 I_s g_0 \frac{T}{T+L}$$

But $g_0 = \frac{\lambda^2}{8\pi t_s \Delta\nu} \Delta N$ using $g(\nu) \approx \frac{1}{\Delta\nu}$

and $I_s = \frac{8\pi t_s h\nu}{\lambda^2 \tau_s} \Delta\nu_H$ homogeneous line width $\Delta\nu_H$

So $P_{out} = \pi w^2 \frac{h\nu}{\tau_s} \frac{\Delta\nu_H}{\Delta\nu} \Delta N \frac{T}{T+L}$

Also, $\Delta N = \frac{R(\tau_2 - \tau_1)}{1 + W_p \tau_2} \approx \frac{R\tau_2}{1+W_p\tau_2}$

and $\tau_s = \frac{\tau_2}{1+W_p\tau_2}$

So $P_{out} = \pi w^2 h\nu \frac{\Delta\nu_H}{\Delta\nu} R \frac{T}{T+L}$

If $\Delta\nu$ + $\Delta\nu_H$ don't depend on $t_s$, then

$P_{out}$ is $\boxed{\text{independent of } t_s}$

Also, note $\nu = \frac{c}{\lambda}$, and $w \propto \sqrt{\lambda}$ in general.

So, $P_{out}$ is also $\boxed{\text{independent of } \lambda}$

## Note on beam waist:

At any point in cavity, mode parameter is

$$\frac{1}{q} = \frac{1}{B}\left[\frac{D-A}{2} \mp i\sqrt{1-\left(\frac{A+D}{2}\right)^2}\right]$$

for $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ = round trip matrix starting from that point

So beam waist $w$ at that point is given by

$$\frac{\lambda}{\pi w^2} = \mathrm{Im}\,\frac{1}{q} = \pm\frac{1}{B}\sqrt{1-\left(\frac{A+D}{2}\right)^2}$$

Note $A, B, C, D$ are independent of $\lambda$.

So, have $\frac{\lambda}{w^2}$ = constant

$$\boxed{w^2 \propto \lambda}$$