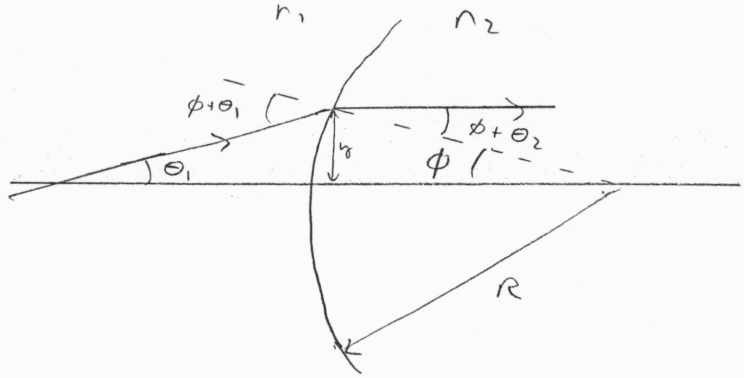
Phys 532 Assignment $\underline{1}$ Solutions

1. a)



Clearly $y_2 = y_1$, so $A = 1$, $B = 0$

Need to calculate θ_2

Angle of normal to surface is $\phi = \frac{y}{R}$

Then angle of incidence is $\phi + \theta_1$

angle of refraction is $\phi + \theta_2$

So $$n_1 \sin(\phi + \theta_1) = n_2 \sin(\phi + \theta_2)$$

In paraxial limit $\phi, \theta_1, \theta_2 \ll 1$ so

$$n_1 \phi + n_1 \theta_1 = n_2 \phi + n_2 \theta_2$$

$$\theta_2 = \frac{n_1 - n_2}{n_2} \phi + \frac{n_1}{n_2} \theta_2$$

$$= \frac{n_1 - n_2}{n_2} \frac{y}{R} + \frac{n_1}{n_2} \theta_2$$

So $C = \frac{n_1 - n_2}{n_2 R}$ $D = \frac{n_1}{n_2}$

$$\boxed{M = \begin{bmatrix} 1 & 0 \\ -\frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}}$$

b) Total matrix is

$$M = M_{R_2} M_{R_1}$$

Take $n = 1$ outside lens
$n = n$ inside

$$= \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1-n}{nR_1} & \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} + \frac{1-n}{R_1} & 1 \end{bmatrix}$$

Compare to matrix for lens: $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

So $\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$

2. a) Starting just before mirror R_1

$M_{TOT} =$

$$\begin{bmatrix} 1 & 570 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 660 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{2}{150} & 1 \end{bmatrix}\begin{bmatrix} 1 & 220 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{2}{100} & 1 \end{bmatrix}\begin{bmatrix} 1 & 130 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{2}{100} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1.256 & 1450.8\,mm \\ -0.0019\,mm^{-1} & 1.36 \end{bmatrix}$$

Using Matlab

Cavity is stable if $\frac{|A+D|}{2} < 1$

here $\boxed{\frac{|A+D|}{2} = 0.052}$, so it is stable

b) For variable R, have

$$M_{TOT} = \begin{bmatrix} 1 & 570 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ -\frac{2}{100} & 1 \end{bmatrix}\begin{bmatrix} 1 & 130 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 27.76 & 1450.8\text{ mm} \\ 0.0253\text{ mm}^{-1} & 1.36 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

Define $= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$

$$= \begin{bmatrix} \alpha + \frac{2\beta}{R} & \beta \\ \gamma + \frac{2\delta}{R} & \delta \end{bmatrix}$$

So, stability requires $\left|\alpha + \delta + \frac{2\beta}{R}\right| < 2$

One end point: $\alpha + \delta + \frac{2\beta}{R} = -2$

$$R = \frac{-2\beta}{\alpha + \delta + 2} = -93.2\text{ mm}$$

Other: $\alpha + \delta + \frac{2\beta}{R} = 2$

$$R = \frac{2\beta}{2 - \alpha - \delta} = -107\text{ mm}$$

So, need $\boxed{-107\text{ mm} < R < -93.2\text{ mm}}$

3. For a stable trajectory, have

$$y_m = y_{max} \sin(m\phi + \phi_{oy})$$

and $$x_m = x_{max} \sin(m\phi - \phi_{ox})$$

where (x_m, y_m) is position on mirror at n^{th} bounce,

and $$\phi = \cos^{-1} \frac{A+D}{2}$$

for single-pass matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

For one full pass of the cavity,

$$M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{2d}{R} & d \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 + \frac{2d}{R} & d \\ \frac{2}{R} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(1 + \frac{2d}{R}\right)^2 + \frac{2d}{R} & \left(1 + \frac{2d}{R}\right) d + d \\ \left(1 + \frac{2d}{R}\right)\left(\frac{2}{R}\right) + \frac{2}{R} & \frac{2d}{R} + 1 \end{bmatrix}$$

So $$\frac{A+D}{2} = \frac{1}{2}\left[\left(1 + \frac{2d}{R}\right)^2 + \frac{2d}{R} + 1 + \frac{2d}{R} \right]$$

$$= b = 1 + \frac{4d}{R} + \frac{2d^2}{R^2}$$

So be periodic, need $m\phi = 2\pi q$

for some integers m & q

Here we want $m = 10$, so beam exits on 10^{th} bounce

Also need q relatively prime to m, otherwise beam will exit early:

For example, if $q = 2$, then also have

$$5\phi = 2\pi$$

and beam retraces path after 5 bounces

Rules out q: 0, 2, 4, 5, 6, 8, 10 etc

Also, can take $0 \le \phi < \pi$, since $\phi = \cos^{-1}\frac{A+D}{2}$

So, require $q < \frac{m}{2} = 5$

This leaves $q = 3$ and $q = 1$

$q = 3$: $\phi = 2\pi \times \frac{3}{10}$

$$\cos\phi = -0.309 = b$$

So
$$1 + \frac{4d}{R} + \frac{2d^2}{R^2} = b \Rightarrow \frac{1}{R^2}(2d^2) + \frac{1}{R}(4d) + (1-b) = 0$$

$$\frac{1}{R} = \frac{1}{4d^2}\left[-4d \pm \sqrt{16d^2 - 8d^2(1-b)}\right]$$

$$= \frac{1}{d}\left[-1 \pm \sqrt{1 - \tfrac{1}{2}(1-b)}\right] = \frac{1}{2}\left[-1 \pm \sqrt{\frac{b+1}{2}}\right]$$

Here $\frac{1}{R} = \frac{1}{d}\begin{cases}-1.588\\-0.412\end{cases}$ so $R = \begin{matrix}-12.6\text{ cm}\\-48.5\text{ cm}\end{matrix}$

Also have $q = 1$

$$\phi = 2\pi \times \frac{1}{10}$$

$$b = \cos\phi = 0.809$$

Still $$\frac{1}{R} = \frac{1}{d}\left[-1 \pm \sqrt{\frac{1+b}{2}}\right]$$

$$= \frac{1}{d}\begin{bmatrix}-0.0489 \\ -1.9511\end{bmatrix}$$

$$R = -409\text{ cm}$$

or -10.25 cm

So, possible R's are $\boxed{-10.25\text{ cm},\ -12.6\text{ cm},\ -48.5\text{ cm},\ -409\text{ cm}}$

4. Peak intensity $$I_0 = \frac{2P}{\pi\, W(z)^2}$$

$$W(z)^2 = W_0^2\left(1 + \frac{z^2}{z_0^2}\right)$$

$$z_0 = \frac{\pi W_0^2}{\lambda}$$

So $$I_0(d) = \frac{2P}{\pi}\,\frac{1}{W_0^2 + \frac{\lambda^2 d^2}{\pi^2 W_0^2}}$$

Find optimum W_0:

$$\frac{dI_0}{dW_0} = \frac{2P}{\pi}(-1)\left(W_0^2 + \frac{\lambda^2 d^2}{\pi^2 W_0^2}\right)^{-2}\left(2W_0 - 2\frac{\lambda^2 d^2}{\pi^2 W_0^3}\right) = 0$$

So $\quad \omega_0^4 = \dfrac{\lambda^2 d^2}{\pi^2}$

$$\boxed{\omega_0 = \sqrt{\frac{\lambda d}{\pi}}}$$

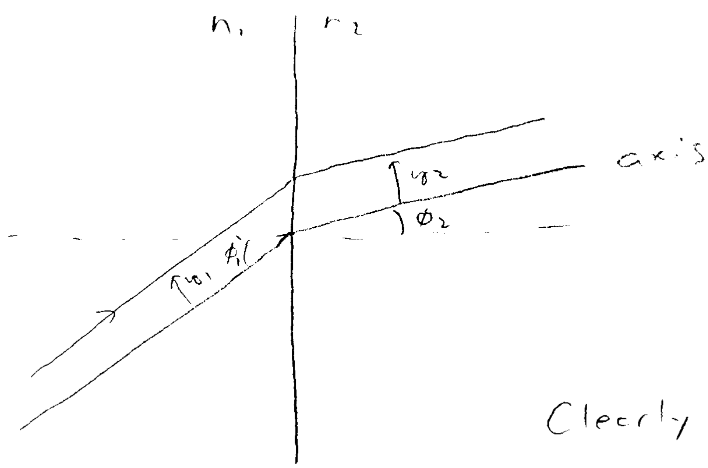
In other words, want to make $d = \dfrac{\pi \omega_0^2}{\lambda} = z_0$

b) $\quad \lambda = 532\,nm$

$$\boxed{\begin{array}{lll} d = 1\,cm & \Rightarrow & \omega_0 = 41\,\mu m \\ 1\,m & \Rightarrow & \omega_0 = 0.41\,mm \\ 100\,m & \Rightarrow & \omega_0 = 4.1\,mm \end{array}}$$

5. First find A & C: Take $\Theta_1 = 0$

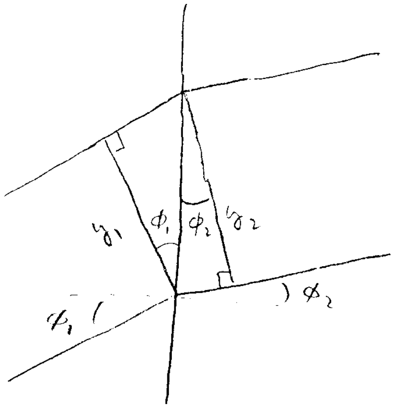
So picture is:



Clearly $\Theta_2 = 0$

so $C = 0$

To get y_2:



See that

$$d = \frac{y_1}{\cos\phi_1} = \frac{y_2}{\cos\phi_2}$$

$$y_2 = \frac{\cos\phi_2}{\cos\phi_1}\, y_1$$

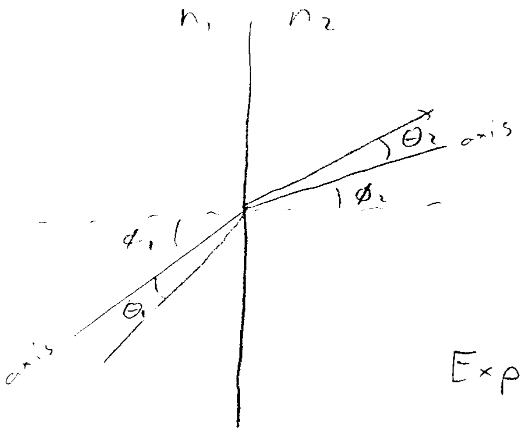
where $\cos\phi_2 = \sqrt{1-\sin^2\phi_2}$

$$\cos\phi_2 = \sqrt{1 - \frac{n_1^2}{n_2^2}\sin^2\phi_1}$$

So $A = \dfrac{\cos\phi_2}{\cos\phi_1}$

Then find B & D: Take $y_1 = 0$

Picture is



Clearly $y_2 = 0$

so $B = 0$

Then have

$$n_1 \sin(\phi_1 + \theta_1) = n_2 \sin(\phi_2 + \theta_2)$$

Expand for small θ:

$$n_1 \sin\phi_1 + \theta_1 n_1 \cos\phi_1 = n_2 \sin\phi_2 + \theta_2 n_2 \cos\phi_2$$

Sine terms cancel, so

$$\Theta_2 = \frac{n_1 \cos\phi_1}{n_2 \cos\phi_2}\,\Theta_1 = D\,\Theta_1$$

Finally,

$$M = \begin{bmatrix} \dfrac{\cos\phi_2}{\cos\phi_1} & 0 \\ 0 & \dfrac{n_1}{n_2}\,\dfrac{\cos\phi_1}{\cos\phi_2} \end{bmatrix}$$