Phys 532 Assignment 10 Solutions

1. a)

$$F = ma = m\ddot{x} = -\frac{dU}{dx} - eE$$

so

$$\boxed{m\ddot{x} + U_0 \frac{x}{a^2} + U_0 \frac{x^2}{a^3} = -e E_0 e^{i\omega t}}$$

If x is very small, neglect x^2 term, and

$$\ddot{x} + \frac{U_0}{ma^2} x = -\frac{eE_0}{m} e^{i\omega t}$$

form $\ddot{x} + \omega_0^2 x = -\frac{eE_0}{m} e^{i\omega t}$

$$\omega_0^2 = \frac{U_0}{ma^2}$$

$$\boxed{\omega_0 = \sqrt{\frac{U_0}{ma^2}}}$$

b) Have

$$\ddot{x} = -\omega^2 x_1 e^{i\omega t} - 4\omega^2 x_2 e^{2i\omega t} - 9\omega^2 x_3 e^{3i\omega t}$$

$$\omega_0^2 x = \omega_0^2 x_1 e^{i\omega t} + \omega_0^2 x_2 e^{2i\omega t} + \omega_0^2 x_3 e^{3i\omega t}$$

$$x^2 = x_1^2 e^{2i\omega t} + x_2^2 e^{4i\omega t} + x_3^2 e^{6i\omega t} + 2x_1x_2 e^{3i\omega t} + 2x_1x_3 e^{4i\omega t} + 2x_2x_3 e^{5i\omega t}$$

Ignore terms higher than $e^{3i\omega t}$, since we don't care about their amplitudes.

$$x^2 \to x_1^2 e^{2i\omega t} + 2x_1x_2 e^{3i\omega t}$$

So equation of motion becomes

$$-\omega^2 x_1 e^{i\omega t} - 4\omega^2 x_2 e^{2i\omega t} - 9\omega^2 x_3 e^{3i\omega t}$$
$$+ \omega_0^2 x_1 e^{i\omega t} + \omega_0^2 x_2 e^{2i\omega t} + \omega_0^2 x_3 e^{3i\omega t}$$
$$+ \beta x_1^2 e^{2i\omega t} + 2\beta x_1 x_2 e^{3i\omega t} = -\frac{e}{m} E_0 e^{i\omega t}$$

$$\beta = \frac{U_0}{ma^3} = \frac{\omega_0^2}{a}$$

Coefficients of each $e^{in\omega t}$ factor are independent, so

(2)

(1) $$-\omega^2 x_1 + \omega_0^2 x_1 = -\frac{e}{m} E_0$$

(2) $$-4\omega^2 x_2 + \omega_0^2 x_2 + \beta x_1^2 = 0$$

(3) $$-9\omega^2 x_3 + \omega_0^2 x_3 + 2\beta x_1 x_2 = 0$$

From (1), $$\boxed{x_1 = \frac{e}{m} E_0 \frac{1}{\omega^2 - \omega_0^2}}$$

From (2), $$x_2 = \frac{\beta}{4\omega^2 - \omega_0^2} x_1^2 = \boxed{\frac{e^2}{m^2} E_0^2 \frac{\beta}{(4\omega^2-\omega_0^2)(\omega^2-\omega_0^2)^2}}$$

From (3), $$x_3 = \frac{2\beta}{9\omega^2 - \omega_0^2} x_1 x_2$$

$$\boxed{x_3 = \frac{e^3 E_0^3}{m^3} \frac{2\beta^2}{(9\omega^2-\omega_0^2)(4\omega^2-\omega_0^2)(\omega^2-\omega_0^2)^3}}$$

c) If $\omega \ll \omega_0$

$$x_1 \approx -\frac{e}{m} E_0 \frac{1}{\omega_0^2} = -\frac{e}{m} E_0 \frac{ma^2}{V_0} = -eE_0 \frac{a^2}{V_0}$$

$$x_2 = -\frac{e^2}{m^2} E_0^2 \frac{\beta}{\omega_0^6} = -\frac{e^2}{m^2} E_0^2 \frac{V_0}{ma^3} \frac{a^6 m^3}{V_0^3}$$

$$= -e^2 E_0^2 \frac{a^3}{V_0^2}$$

$$x_3 = \frac{e^3 E_0^3}{m^3} \frac{2\beta^2}{\omega_0^{10}} = 2 \frac{e^3 E_0^3}{m^3} \frac{V_0^2}{m^2 a^6} \frac{m^5 a^{10}}{V_0^5}$$

$$= 2 e^3 E_0^3 \frac{a^4}{V_0^3}$$

So

$$P = -exN$$

$$= Ne^2 E \frac{a^2}{V_0} + Ne^3 E^2 \frac{a^3}{V_0^2} - 2N'e^4 E^3 \frac{a^4}{V_0^3}$$

$$\equiv \varepsilon_0 \chi E + 2d E^2 + 4\chi^{(3)} E^3$$

So $\boxed{\chi = N \frac{e^2 a^2}{\varepsilon_0 V_0}}$

$\boxed{d = \frac{1}{2} N \frac{e^3 a^3}{V_0^2}}$

$\boxed{\chi^{(3)} = -\frac{1}{2} N \frac{e^4 a^4}{V_0^3}}$

Can write

$$P = Nea \left\{ \frac{eEa}{V_0} + \left(\frac{eEa}{V_0}\right)^2 - 2\left(\frac{eEa}{V_0}\right)^3 \right\}$$

So if $eE \ll \frac{V_0}{a}$, terms are decreasing.

2. We have

$$P(2\omega_1) = d\, E(\omega_1)^2$$
$$P(2\omega_2) = d\, E(\omega_2)^2$$
$$P(\omega_1+\omega_2) = 2d\, E(\omega_1)E(\omega_2)$$

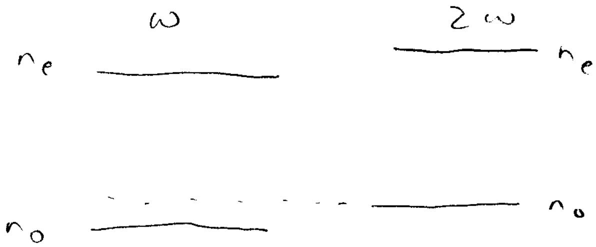
So if these components can't be resolved, have

$$P_{TOT} = P(2\omega_1) + P(2\omega_2) + P(\omega_1+\omega_2)$$

$$= d\left\{ E(\omega_1)^2 + E(\omega_2)^2 + 2E(\omega_1)E(\omega_2) \right\}$$

$\boxed{P = d\left[E(\omega_1) + E(\omega_2)\right]^2}$ as desired

3. By tilting crystal, can adjust n_e' between n_o and n_e



So want to use
$$n_o(2\omega) = n_e'(\omega)$$

So
$$\frac{\cos^2\theta}{n_o(\omega)^2} + \frac{\sin^2\theta}{n_e(\omega)^2} = \frac{1}{n_o(2\omega)^2} \qquad \cos^2\theta = 1-\sin^2\theta$$

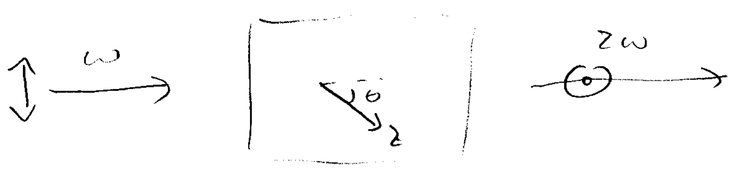
$$\sin^2\theta\left[\frac{1}{n_e(\omega)^2} - \frac{1}{n_o(\omega)^2}\right] = \frac{1}{n_o(2\omega)^2} - \frac{1}{n_o(\omega)^2}$$

$$\sin^2\theta = \frac{\dfrac{1}{n_o(2\omega)^2} - \dfrac{1}{n_o(\omega)^2}}{\dfrac{1}{n_e(\omega)^2} - \dfrac{1}{n_o(\omega)^2}} = \frac{\dfrac{1}{4.856^2} - \dfrac{1}{4.794^2}}{\dfrac{1}{6.243^2} - \dfrac{1}{4.794^2}}$$

$$= 0.0618$$

$$\boxed{\theta = 14.4^\circ}$$

So we have extra-ordinary beam at ω, and ordinary beam at 2ω

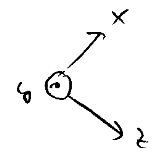


Need to specify x, y

Coupling constants are
$$d_{11} = d_{xxx} \;:\; E_x(2\omega) \propto E_x(\omega)^2$$
$$d_{12} = d_{xyy} \;:\; E_x(2\omega) \propto E_y(\omega)^2$$
$$d_{26} = d_{yxy} \;:\; E_y(2\omega) \propto 2E_x(\omega)E_y(\omega)$$

Check simple cases:

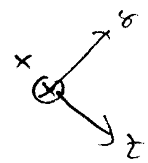
if crystal axes are



Have $E_x(\omega)$ & $E_y(2\omega)$

doesn't work.

if axes are

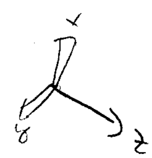


have $E_y(\omega) = \cos\theta\, E(\omega)$ and $E_x(2\omega) = E(2\omega)$

works with d_{12}:

$$d' = \underbrace{\tfrac{1}{2}}_{(\omega_1 = \omega_2)} d_{12} \cos^2\theta$$

Just to be sure, check for x at angle ϕ out of page:



Take $E(\omega) = \uparrow = E_{x'}(\omega)$

$E(2\omega) = \odot = E_{y'}(2\omega)$

Then
$$E_x(\omega) = \cos\phi \cos\theta\, E(\omega)$$
$$E_y(\omega) = -\sin\phi \cos\theta\, E(\omega)$$
$$P_{y'}(2\omega) = \cos\phi\, P_y(2\omega) + \sin\phi\, P_x(2\omega)$$

Then
$$d' = \overset{\omega_1 = \omega_2 \text{ factor}}{\tfrac{1}{2}} \Big[d_{xxx} \left(\sin\phi \cdot \cos^2\phi \cos^2\theta \right) + d_{xyy} \left(\sin\phi \sin^2\phi \cos^2\theta \right) + 2 d_{yxy} \left(-\cos^2\phi \sin\phi \cos^2\theta \right) \Big]$$

$(d_{12} = -d_{11})$

$(d_{26} = -d_{11})$

Or $d' = \frac{1}{2} d_{11} \sin\phi \cos^2\Theta \left[\cos^2\phi - \sin^2\phi + 2\cos^2\phi \right]$

$= \frac{1}{2} d_{11} \sin\phi \cos^2\Theta \,(4\cos^2\phi - 1)$

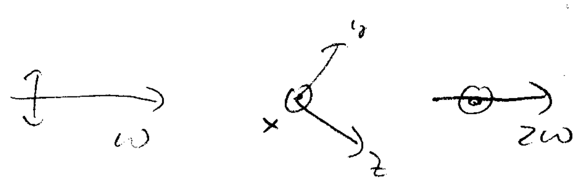
$$\sin\phi\,(4\cos^2\phi - 1) = 2\sin\phi\cos^2\phi + \sin\phi\,(2\cos^2\phi - 1)$$
$$= \sin 2\phi \cos\phi + \sin\phi \cos 2\phi$$
$$= \sin 3\phi$$

So $d' = \frac{1}{2} d_{11} \cos^2\Theta \sin 3\phi$

Best at $\phi = \pm 60^\circ, \pm 90^\circ$

Best case, $|d'| = \frac{1}{2} d_{11} \cos^2\Theta = 2.67 \times 10^{-21} \frac{C}{V^2}$

Simplest is $\phi = 90^\circ$, as originally considered



4. From Dmitriev, Gurzadyan + Nikogosyan & Yariv

Symmetry class 3m, uniaxial. $d_{22} = -d_{21} = -d_{16}$, $d_{24} = d_{15} \approx d_{31} = d_{32}$, d_{33}

Transmission range 200 nm to 3.5 μm

$d_{22} = 2 \times 10^{-23} \frac{C}{V^2}$

$d_{31} = 1.4 \times 10^{-24} \frac{C}{V^2}$

$d_{33} = ?$

λ	n_o	n_e
532 nm	1.6755	1.5549
1064 nm	1.6551	1.5425

5. Have $\vec{E}(t) = \vec{E}(0) + \mathrm{Re}\,\vec{E}(\omega)e^{i\omega t} = \vec{E}(0) + \frac{1}{2}\vec{E}(\omega)e^{i\omega t} + \frac{1}{2}\vec{E}^*(\omega)e^{-i\omega t}$

$$P_i(t) = \sum_{jk} 2d_{ijk}\, E_j(t)\, E_k(t)$$

$$= \sum_{jk} 2d_{ijk} \left(E_j(0) + \tfrac{1}{2}E_j(\omega)e^{i\omega t} + \tfrac{1}{2}E_j(\omega)^* e^{-i\omega t}\right) \times \left(E_k(0) + \tfrac{1}{2}E_k(\omega)e^{i\omega t} + \tfrac{1}{2}E_k(\omega)^* e^{-i\omega t}\right)$$

Component of P_i at ω is

$$\tfrac{1}{2}\left[P_i(\omega)e^{i\omega t} + P_i^*(\omega)e^{-i\omega t}\right]$$
$$= \sum_{jk} 2d_{ijk}\left[\tfrac{1}{2}E_j(0)E_k(\omega)e^{i\omega t} + \tfrac{1}{2}E_j(\omega)E_k(0)e^{i\omega t} + \tfrac{1}{2}E_j(0)E_k^*(\omega)e^{-i\omega t} + \tfrac{1}{2}E_j^*(\omega)E_k(0)e^{-i\omega t}\right]$$

So $P_i(\omega) = \sum_{jk} 2d_{ijk}\left[E_j(0)E_k(\omega) + E_j(\omega)E_k(0)\right]$

$$= \sum_{jk} 4d_{ijk}\, E_j(\omega)\, E_k(0)$$

Since $d_{ijk} = d_{ikj}$

So if we define $\Delta\chi_{ij} \equiv \frac{1}{\varepsilon_0}\sum_k 4d_{ijk}E_k(0)$, have contribution

$$P_i(\omega) = \varepsilon_0 \sum_j \Delta\chi_{ij}\, E_j(\omega)$$

So linear susceptibility is changed by $\Delta\chi_{ij}$

Now r_{ijk} defined by

$$\Delta\eta_{ij} = \sum_k r_{ijk}\, E_k(0)$$

So we need to relate Δx_{ij} to $\Delta \eta_{ij}$

Have $\bar{\eta} = \varepsilon_0 \bar{\varepsilon}^{-1}$

$\bar{\varepsilon} = \bar{1} + \bar{x}$

So $\bar{\eta} = \varepsilon_0 (\bar{1} + \bar{x})^{-1}$

In general, if $B = A^{-1}$
and A is modified by ΔA, how does B change?

Have $AB = 1$

So $(A + \Delta A)(B + \Delta B) = 1$

$AB + (\Delta A)B + A(\Delta B) + (\Delta A)(\Delta B) = 1$

Assume $(\Delta A)(\Delta B)$ is negligibly small
Also, $AB = 1$, so

$(\Delta A)B + A(\Delta B) = 0$

$\Delta B = -A^{-1}(\Delta A)B$

$\Delta B = -B(\Delta A)B$

So here $\Delta \eta = -\eta \, \Delta x \, \eta$

or $\Delta \eta_{ij} = -\sum_{kl} \eta_{ik} \Delta x_{kl} \eta_{lj}$

So $\sum_m r_{ijm} E_m(0) = -\sum_{kl} \eta_{ik} \frac{1}{\varepsilon_0} \sum_m 4 d_{klm} E_m(0) \eta_{lj}$

$\sum_m \left[r_{ijm} + \frac{4}{\varepsilon_0} \sum_{kl} \eta_{ik} d_{klm} \eta_{lj} \right] E_m(0) = 0$

Or

$$r_{ijm} = -\frac{4}{\varepsilon_0} \sum_{kl} \eta_{ik} d_{klm} \eta_{lj}$$

But for unmodified crystal, η is diagonal

$$\eta_{ii} = \frac{\varepsilon_0}{\varepsilon_{ii}}$$

So need $k = i$ and $l = j$

$$r_{ijm} = -\frac{4}{\varepsilon_0} \frac{\varepsilon_0}{\varepsilon_{ii}} d_{ijm} \frac{\varepsilon_0}{\varepsilon_{jj}}$$

Or

$$\boxed{r_{ijk} = -4\varepsilon_0 \frac{d_{ijk}}{\varepsilon_{ii}\,\varepsilon_{jj}}}$$