Phys 532 Assignment 2 Solutions

1. Have $w_1^2 = w_0^2\left(1 + \frac{z_1^2}{z_0^2}\right) = w_0^2 + \frac{\lambda^2 z_1^2}{\pi^2 w_0^2}$

where w_0 = waist at focus

z_1 = distance from w_1 to focus

are to be determined

Also, $w_2^2 = w_0^2 + \frac{\lambda^2}{\pi^2 w_0^2}(z_1 + d)^2$ $d = 0.1\,m$ (take $d > 0$)

Rewrite to simplify: define $u_0 = \frac{\pi}{\lambda} w_0^2$ $(= z_0)$

$u_1 = \frac{\pi}{\lambda} w_1^2 = 0.8551\,m$

$u_2 = \frac{\pi}{\lambda} w_2^2 = 3.3842\,m$

* Note, since u_1 & u_2 both much larger than d, can see that $z_0 \ll d$ and use large z limit. Easier solution, but I'll work through general case.

So

$$u_1 = u_0 + \frac{z_1^2}{u_0}$$

$$u_2 = u_0 + \frac{1}{u_0}(z_1 + d)^2$$

$$= u_0 + \frac{1}{u_0}(z_1^2 + 2z_1 d + d^2)$$

$$= u_1 + \frac{1}{u_0}(2z_1 d + d^2)$$

So $$z_1 = \frac{u_0(u_2 - u_1) - d^2}{2d}$$

Substitute:

$$u_1 = u_0 + \frac{1}{u_0}\left[\frac{u_0(u_2-u_1)-d^2}{2d}\right]^2$$

$$= u_0 + \frac{1}{4u_0 d^2}\left[u_0^2(u_2-u_1)^2 - 2d^2 u_0(u_2-u_1) + d^4\right]$$

$$u_1 = \left[1 + \frac{(u_2-u_1)^2}{4d^2}\right]u_0 - \frac{1}{2}(u_2-u_1) + \frac{d^2}{4u_0}$$

$$0 = \left[4 + \frac{(u_2-u_1)^2}{d^2}\right]u_0^2 - 2(u_1+u_2)u_0 + d^2$$

Solve quadratic:

$$u_0 = \frac{1}{2\left[4+\frac{(u_2-u_1)^2}{d^2}\right]}\left[2(u_1+u_2) \pm \sqrt{4(u_1+u_2)^2 - 4d^2\left(4 + \frac{u_2-u_1}{d^2}\right)}\right]$$

$$= \frac{d^2}{(u_2-u_1)^2+4d^2}\left[u_1+u_2 \pm \sqrt{(u_1+u_2)^2 - (u_1-u_2)^2 - 4d^2}\right]$$

$$\boxed{u_0 = \frac{d^2}{(u_2-u_1)^2+4d^2}\left[u_1+u_2 \pm \sqrt{u_1 u_2 - d^2}\right]}$$

Evaluate, get $u_0 = z_0 = 11.9$ mm or 1.31 mm

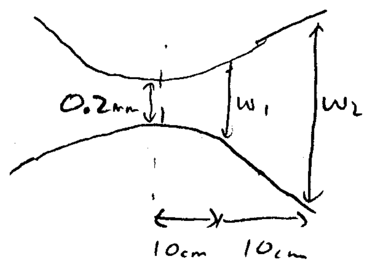
$$w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \boxed{0.2 \text{ mm or } 66.5\,\mu\text{m}}$$

Then

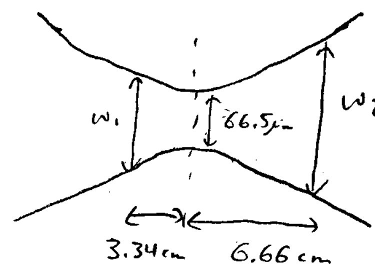
$$z_1 = \frac{u_0(u_2-u_1)-d^2}{2d} = \boxed{10 \text{ cm or } -3.34 \text{ cm}}$$

Both solutions are valid:

a):



b)



2. Have $I(x,y) = \dfrac{2P_0}{\pi w^2} e^{-\frac{2(x^2+y^2)}{w^2}}$ x, y = distance from center

w = width to be measured

P_0 = total power

With razor blade at x, transmit

$$P(x) = \int_x^\infty dx' \int_{-\infty}^\infty dy \, I(x', y)$$

Note $\displaystyle\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$

So $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{2y^2}{w^2}} dy = w\sqrt{\frac{\pi}{2}}$

and $$\int_x^\infty dx' e^{-\frac{2x'^2}{w^2}} = \int_0^\infty e^{-\frac{2x'^2}{w^2}} dx' - \int_0^x e^{-\frac{2x'^2}{w^2}} dx'$$

$$= \frac{1}{2} w\sqrt{\frac{\pi}{2}} - \frac{w}{\sqrt{2}} \int_0^{\frac{x\sqrt{2}}{w}} e^{-u^2} du$$

$$= \frac{1}{2} w\sqrt{\frac{\pi}{2}} - \frac{w}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x\sqrt{2}}{w}\right)$$

So $$P(x) = \frac{2P_0}{\pi w^2} \times w\sqrt{\frac{\pi}{2}} \times \frac{1}{2} w \sqrt{\frac{\pi}{2}} \left[1 - \mathrm{erf}\left(\frac{x\sqrt{2}}{w}\right)\right]$$

$$= \frac{P_0}{2}\left[1 - \mathrm{erf}\left(\frac{x\sqrt{2}}{w}\right)\right]$$

So $$\frac{P(x)}{P_0} = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{x\sqrt{2}}{w}\right)\right]$$

At x_1, $x = x_1 - x_0$ and $\frac{P(x)}{P_0} = 0.7$

So $1 - \mathrm{erf}\left(\frac{(x_1 - x_0)\sqrt{2}}{w}\right) = 1.4$

$\mathrm{erf}\left(\frac{(x_1 - x_0)\sqrt{2}}{w}\right) = -0.4$

so $\mathrm{erf}\left[\frac{(x_0 - x_1)\sqrt{2}}{w}\right] = +0.4$

From tables, $\mathrm{erf}(0.37) = 0.4$

So $$\frac{(x_0 - x_1)\sqrt{2}}{w} = 0.37$$

At x_2, $\frac{P(x)}{P_0} = 0.3$

So $1 - \mathrm{erf}\left[\frac{(x_2 - x_0)\sqrt{2}}{w}\right] = 0.6$

$\mathrm{erf}\left[\frac{(x_2 - x_0)\sqrt{2}}{w}\right] = 0.4$

so $$\frac{(x_2 - x_0)\sqrt{2}}{w} = 0.37$$

Adding equations, get

$$\frac{(x_2-x_1)\sqrt{2}}{w} = 2\times 0.37$$

or $\quad x_2 - x_1 = 0.37\times\sqrt{2}\,w = 0.523\,w$

So $\quad \boxed{w = \frac{x_2-x_1}{0.523} = 1.91\,\Delta}$

3. Let q_0 = input value

q_1 = value after system 1

q_2 = value after system 2

Then $\quad q_1 = \frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}$

$$q_2 = \frac{A_2 q_1 + B_2}{C_2 q_1 + D_2} = \frac{A_2\left(\frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}\right) + B_2}{C_2\left(\frac{A_1 q_0 + B_1}{C_1 q_0 + D_1}\right) + D_2}$$

$$= \frac{(A_1A_2 + C_1B_2)q_0 + (B_1A_2 + D_1B_2)}{(A_1C_2 + C_1D_2)q_0 + (B_1C_2 + D_1D_2)}$$

While

$$\begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_1A_2 + C_1B_2 & B_1A_2 + D_1B_2 \\ A_1C_2 + C_1D_2 & B_1C_2 + D_1D_2 \end{bmatrix}$$

So, $\boxed{q_2 = \dfrac{A_3 q_0 + B_3}{C_3 q_0 + D_3}}$ as desired

4. Input beam has $z = 35\,mm$

$$z_0 = \frac{\pi w_0^2}{\lambda} = 14.76\,mm$$

so $q_1 = 35 + i\,14.76\ mm$

Lens matrix is $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

so after lens, $q_2 = \dfrac{35 + i\,14.76}{-\frac{1}{25}(35 + i\,14.76) + 1}$

$$= -44.7 + i\,29\ mm = z' + i z_0'$$

So, resulting focus is $\boxed{44.7\,mm}$ from lens

with waist $w_0 = \sqrt{\dfrac{\lambda z_0'}{\pi}} = \boxed{70\,\mu m}$

Ray optics:

i) Collimated input gives focus at $z = f = \boxed{25\,mm}$

ii) Use $\dfrac{1}{s'} = \dfrac{1}{f} - \dfrac{1}{s} = \dfrac{1}{25\,mm} - \dfrac{1}{35\,mm} = \dfrac{1}{87.5\,mm}$

Focus at $\boxed{87.5\,mm}$

Gaussian beam focus is in between

5. At mirror, input beam has $z = 100\,\text{mm}$

$$z_0 = \frac{\pi w_0^2}{\lambda} = 187.5\,\text{mm}$$

Mirror has matrix $\begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$

So, reflected beam has

$$q' = \frac{z + iz_0}{\frac{2}{R}(z+iz_0)+1}$$

Want R such that $q' = -z + iz_0$

(now $-z$, because converging instead of diverging)

So

$$\frac{2}{R}(z+iz_0)+1 = \frac{z+iz_0}{-z+iz_0}$$

$$\frac{2}{R} + \frac{1}{z+iz_0} = \frac{1}{-z+iz_0}$$

$$\frac{2}{R} = \frac{1}{-z+iz_0} - \frac{1}{z+iz_0} = \frac{(z+iz_0)-(-z+iz_0)}{-(z^2+z_0^2)}$$

$$= \frac{-2z}{z^2+z_0^2}$$

$$R = -\frac{z^2+z_0^2}{z}$$

$$= -\frac{100\,\text{mm}^2 + 187.5\,\text{mm}^2}{100\,\text{mm}}$$

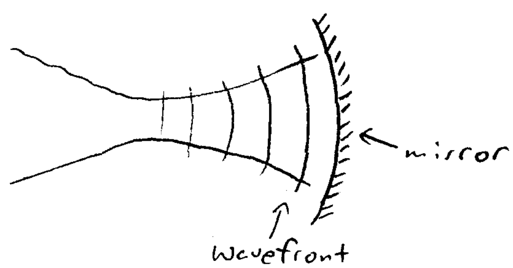
$$\boxed{R = -451\,\text{mm}}$$

At the mirror, beam has curvature

$$\frac{1}{R_b} = \frac{z}{z^2 + z_0^2}$$

So $\boxed{R_{mirror} = -R_{beam}}$

Following our sign convention, mirror & wavefront are matched:



6. a) If $A = f(x,y)\, N(z)\, e^{-\alpha(z)\rho^2}$

then $\frac{\partial A}{\partial z} = f \frac{\partial N}{\partial z} e^{-\alpha\rho^2} + fN\left(-\frac{\partial \alpha}{\partial z}\rho^2\right) e^{-\alpha\rho^2}$

$$= \left(\frac{N'}{N} - \alpha'\rho^2\right) A$$

So paraxial wave equation becomes

$$\nabla_T^2 A - 2ik\left(\frac{N'}{N} - \alpha'\rho^2\right)A = 0$$

$$\nabla_T^2 A + 2ik\alpha'\rho^2 A = 2ik\frac{N'}{N} A$$

Has form of Schrodinger equation, for 2D harmonic oscillator,

$$-\frac{\hbar^2}{2m}\nabla_T^2 \psi + \frac{1}{2}m\omega^2\rho^2\psi = E\psi$$

with $\frac{m^2\omega^2}{\hbar^2} \leftrightarrow -2ik\alpha'$

$$\frac{2mE}{\hbar^2} \leftrightarrow -2ik\frac{N'}{N}$$

b) Know that width of wave function $\Delta x \propto \sqrt{n+\frac{1}{2}}$ for state $|n\rangle$

[To prove: $\langle H\rangle = E = \hbar\omega(n+\frac{1}{2}) = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle$

by virial theorem, $\frac{1}{2m}\langle p^2\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$

So $\frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{2}\hbar\omega(n+\frac{1}{2})$

$$\langle x^2\rangle = \frac{\hbar}{m\omega}(n+\frac{1}{2})$$

$$\Delta x = \sqrt{\langle x^2\rangle} = \sqrt{\frac{\hbar}{m\omega}(n+\frac{1}{2})}$$]

Also true for Hermite-Gaussian beam, so expect

$$\boxed{W_\ell \propto \sqrt{\ell+\frac{1}{2}}}$$