Phys 532 Assignment 3 Solutions

1. a) From HW I, have round trip matrix

$$\begin{bmatrix} -1.256 & 1.4508\,m \\ -1.8667\,m^{-1} & 1.36 \end{bmatrix}$$

for round trip starting just before mirror M_1.

The at this point, $q = \frac{1}{C}\left[\frac{A-D}{2} \pm i\sqrt{1-\left(\frac{A+D}{2}\right)^2} \right]$

$= 0.7\,m + i\,0.535\,m$

So for leg 3-4-1, have focus 700 mm behind M_1, or $\boxed{130\ mm}$ behind M_4

[Note mirror M_4 has no effect, since $R_4 = \infty$]

For this focus, $w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \boxed{370\,\mu m}$

For leg 1-2, just propagate q through mirror M_1:

metrix $= \begin{bmatrix} 1 & 0 \\ \frac{2}{R_1} & 1 \end{bmatrix}$ $R_1 = -.1\,m$

So $q' = \dfrac{q}{\frac{2}{R_1}q + 1} = -52.3\,mm + i\,1.887\,mm$

So, get focus $\boxed{52.3\text{ mm}}$ in front of M_1

beam waist $w_0 = \sqrt{\frac{\lambda z_0}{\pi}} = \boxed{21.9\,\mu m}$

Finally, check leg 2-3:

Use matrix $M = \begin{bmatrix} 1 & 0 \\ -\frac{2}{0.1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.13 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0.13\text{ m} \\ -20\text{ m}^{-1} & -1.6 \end{bmatrix}$

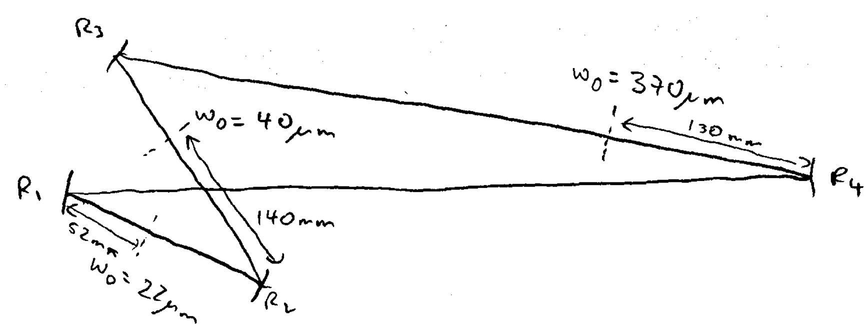
$q' = \dfrac{q + 0.13}{-20q - 1.6}$ for $q = -5.23\times10^{-3} + i\,1.887\times10^{-3}\text{ m}$

$= -139.8 + i\,6.12\text{ mm}$

Focus is $\boxed{139.8\text{ mm}}$ past M_2, with

waist $= \sqrt{\frac{\lambda z_0}{\pi}} = \boxed{39.5\,\mu m}$

Picture:



b) Output beam is diverging from focus with

$w_0 = 370\,\mu m$

So, has divergence angle $\Theta = \frac{\lambda}{\pi w_0} = \boxed{0.7\,mrad}$

Free spectral range $\nu_F = \frac{c}{L}$

$L = 130 + 220 + 660 + 570\,mm = 1.58\,m$

$\boxed{\nu_F = 190\,MHz}$

Guoy phase

les 1-2: $z_1 = -52.3\,mm$ $z_2 = +77.7\,mm$

$z_0 = 1.887\,mm$

$\Delta \zeta_{12} = \tan^{-1}\frac{z_2}{z_0} - \tan^{-1}\frac{z_1}{z_0}$

$= 1.546 + 1.535 = 3.081\,rad$

les 2-3: $z_1 = -139.8\,mm$ $z_2 = 80.2\,mm$

$z_0 = 6.12\,mm$

$\Delta \zeta_{23} = \tan^{-1}\frac{80.2}{6.12} + \tan^{-1}\frac{139.8}{6.12}$

$= 3.022\,rad$

les 3-1: $z_1 = -530\,mm$ $z_2 = 700\,mm$

$z_0 = 535\,mm$

$\Delta \zeta_{31} = \tan^{-1}\frac{700}{535} + \tan^{-1}\frac{530}{535}$

$= 1.699\,rad$

So $\Delta\rho_{TOT} = \Delta\rho_{12} + \Delta\rho_{23} + \Delta\rho_{31}$

$$\boxed{\Delta\rho = 7.802 \text{ rad}}$$

After one pass through cavity, intensity is reduced by factor $(0.995)^3 \times 0.95 = 0.9358$

So, loss per pass $= \boxed{0.0642 = \Gamma}$

Then $\delta\nu = \nu_F \frac{\Gamma}{2\pi} = \boxed{1.94 \text{ MHz}}$

Finesse $\boxed{\mathcal{F} = \frac{\nu_F}{\delta\nu} = \frac{2\pi}{\Gamma} = 97.8}$

$$\boxed{Q = \frac{\nu}{\delta\nu} = \frac{c}{\lambda\,\delta\nu} = 1.93\times10^8}$$

$$\boxed{\tau_p = \frac{1}{2\pi\,\delta\nu} = 82 \text{ ns}}$$

$$\boxed{\alpha = \frac{1}{L} \ln \frac{1}{1-\Gamma} = 0.042 \text{ m}^{-1}}$$

4. a)

$$m\ddot{x} + b\dot{x} + kx = 0$$

Rewrite:

$$\ddot{x} + \nabla\dot{x} + \omega_0^2 x = 0$$

$$\nabla = \frac{b}{m}$$

$$\omega_0^2 = \frac{k}{m}$$

Plug in $x(t)$:

$$\dot{x} = A\left(-\frac{\gamma}{2}\right)e^{-\gamma t/2}\cos(\bar{\omega}t+\phi) - A\bar{\omega}e^{-\gamma t/2}\sin(\bar{\omega}t+\phi)$$

$$\ddot{x} = A\left(\frac{\gamma^2}{4}\right)e^{-\gamma t/2}\cos(\bar{\omega}t+\phi) + 2A\frac{\gamma}{2}\bar{\omega}e^{-\gamma t/2}\sin(\bar{\omega}t+\phi) - A\bar{\omega}^2 e^{-\gamma t/2}\cos(\bar{\omega}t+\phi)$$

So have equation:

$$Ae^{-\gamma t/2}\left[\left(\frac{\gamma^2}{4}-\bar{\omega}^2\right)\cos(\bar{\omega}t+\phi) + \gamma\bar{\omega}\sin(\bar{\omega}t+\phi)\right]$$

$$+ \nabla Ae^{-\gamma t/2}\left[-\frac{\gamma}{2}\cos(\bar{\omega}t+\phi) - \bar{\omega}\sin(\bar{\omega}t+\phi)\right]$$

$$+ \omega_0^2 Ae^{-\gamma t/2}\cos(\bar{\omega}t+\phi) = 0$$

So $Ae^{-\gamma t/2}$ cancels out, set

$$\left(\frac{\gamma^2}{4} - \bar{\omega}^2 + \omega_0^2 - \frac{\gamma}{2}\sigma\right)\cos(\bar{\omega}t + \phi)$$

$$+ \left(\gamma\bar{\omega} - \sigma\bar{\omega}\right)\sin(\bar{\omega}t + \phi) = 0$$

Coefficients of sine and cosine must vanish, so

$$\boxed{\gamma = \sigma = b/m}$$

and

$$\bar{\omega}^2 = \omega_0^2 + \frac{\gamma^2}{4} - \frac{\gamma^2}{2} = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\boxed{\bar{\omega} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

b) Then $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2$

$$= \frac{1}{2}mA^2e^{-\gamma t}\left[-\frac{\gamma}{2}\cos(\bar{\omega}t+\phi) - \bar{\omega}\sin(\bar{\omega}t+\phi)\right]^2$$

$$+ \frac{1}{2}mA^2e^{-\gamma t}\omega_0^2\cos^2(\bar{\omega}t+\phi)$$

$$= \frac{1}{2}mA^2e^{-\gamma t}\left[\frac{\gamma^2}{4}\cos^2(\bar{\omega}t+\phi) + \gamma\bar{\omega}\cos(\bar{\omega}t+\phi)\sin(\bar{\omega}t+\phi)\right.$$
$$\left. + \bar{\omega}^2\sin^2(\bar{\omega}t+\phi) + \omega_0^2\cos^2(\bar{\omega}t+\phi)\right]$$

Use $\bar{\omega}^2 = \omega_0^2 - \frac{\gamma^2}{4}$, and trig identities

$$\boxed{E = \frac{1}{2} m A^2 e^{-\gamma t}\left[\omega_0^2 + \frac{\gamma^2}{4}\cos 2(\bar{\omega}t+\phi) + \frac{\gamma}{2}\bar{\omega}\sin 2(\bar{\omega}t+\phi)\right]}$$

c) If $\gamma \ll \omega$, then $e^{-\gamma t}$ doesn't change much during one cycle, so cos & sin term average out

$$\langle E \rangle \approx \frac{1}{2} m\omega_0^2 A^2 e^{-\gamma t}$$

$$\frac{d\langle E\rangle}{dt} = -\frac{\gamma}{2} m\omega_0^2 A^2 e^{-\gamma t}$$

So $\boxed{Q' = \frac{\bar{\omega}}{\gamma}}$

d) If $F = F_0 e^{i\omega t}$, have

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 \ddot{x} = f_0 e^{i\omega t} \qquad f_0 = \frac{F_0}{m}$$

Try solution $x = B e^{i\omega t}$

Then

$$B(-\omega^2) + \gamma B(i\omega) + \omega_0^2 B = f_0$$

$$\boxed{B = \frac{f_0}{\omega_0^2 - \omega^2 + i\gamma\omega}}$$

e) $$|x|^2 = |B|^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$$

If $\gamma \ll \omega_0$, then resonance occurs very close to $\omega = \omega_0$

So simplify $$\omega_0^2 - \omega^2 = (\omega_0+\omega)(\omega_0-\omega) \approx 2\omega_0(\omega_0-\omega)$$

and $$\gamma^2\omega^2 = \gamma^2\omega_0^2$$

So $$|x|^2 = \frac{f_0^2}{\omega_0^2} \frac{1}{4(\omega-\omega_0)^2 + \gamma^2}$$

$$\text{Peak} = \frac{f_0^2}{\omega_0^2\gamma^2}$$

So half max at $$4(\omega-\omega_0)^2 + \gamma^2 = 2\gamma^2$$

$$(\omega-\omega_0)^2 = \frac{\gamma^2}{4}$$

$$\omega = \omega_0 \pm \frac{\gamma}{2}$$

So FWHM is $\boxed{\Delta\omega = \gamma}$

f) $$Q = \frac{\bar{\omega}}{\Delta\omega} = \frac{\bar{\omega}}{\gamma} = Q'$$ as desired.

5. N photons have energy $\mathcal{E} = Nh\nu$

Modes of symmetric cavity have focus at center,

and $$z_0 = \frac{1}{2}\sqrt{-d(2R+d)}$$

Total field in cavity is sum of forward & backward beams:

$$U(\rho,z) = A_0 \frac{w_0}{w(z)} e^{-ikz} e^{-ik\frac{\rho^2}{2R(z)}} e^{-\rho^2/w(z)^2} e^{i\zeta(z)}$$

$$+ A_0 \frac{w_0}{w(z)} e^{+ikz} e^{-ik\frac{\rho^2}{2R(z)}} e^{-\rho^2/w(z)^2} e^{i\zeta(z)}$$

$$= 2A_0 \cos kz \frac{w_0}{w(z)} e^{-ik\frac{\rho^2}{2R(z)}} e^{-\rho^2/w(z)^2} e^{i\zeta(z)}$$

So $$I(\rho,z) = I_0 \cos^2 kz \frac{w_0^2}{w(z)^2} e^{-2\rho^2/w(z)^2}$$

$$I_0 = \text{peak intensity} = I(0,0)$$

Then total energy is

$$\mathcal{E} = \frac{1}{c}\int_0^\infty 2\pi\rho\, d\rho \int_{-d/2}^{d/2} dz\, I(\rho,z)$$

$$= \frac{I_0 w_0^2}{c}\int_{-d/2}^{d/2} dz \cos^2 kz \frac{1}{w(z)^2}\int_0^\infty 2\pi\rho\, e^{-2\rho^2/w(z)^2} d\rho$$

From book, ρ integral is $\frac{1}{2}\pi w(z)^2$

So $$\mathcal{E} = \frac{I_0 w_0^2}{c}\frac{\pi}{2}\int_{-d/2}^{d/2} \cos^2 kz\, dz$$

Then $\int_{-d/2}^{d/2} \cos^2 kz \, dz = \frac{1}{2} \int_{-d/2}^{d/2} (1 + \cos 2kz) \, dz$

$$= \frac{d}{2} + \frac{1}{2k} \sin kd$$

But for a resonant mode, have $kd = \pi q$ for integer q, leaving

$$\mathcal{E} = \frac{I_0 w_0^2}{c} \frac{\pi d}{4} = N h\nu$$

$$I_0 = \frac{4}{\pi} \frac{N h \nu c}{w_0^2 d}$$

Use $\pi w_0^2 = z_0 \lambda$ and $\nu = \frac{c}{\lambda}$

$$I_0 = 4N \frac{h\nu^2}{z_0 d}$$

Here $\nu = \frac{c}{\lambda} = 6 \times 10^{14} \, s^{-1}$

$$z_0 = \frac{1}{2} \sqrt{-d(2R+d)} = 5 \text{ cm}$$

$$I_0 = 4 \times 10^4 \frac{6.626 \times 10^{-34} \, J \cdot s \cdot (6 \times 10^{14} s^{-1})^2}{5\text{cm} \cdot 10\text{cm}}$$

$$\boxed{I_0 = 0.19 \, W/cm^2}$$

a) Transverse mode spacing is $\frac{\Delta\zeta}{2\pi}\nu_F$

(Book has $\frac{\Delta\zeta}{\pi}$, but their $\Delta\zeta$ is only half of round trip $\Delta\zeta$!)

From problem 2, $\Delta\zeta = 7.802$

So $\frac{\Delta\zeta}{2\pi} = 1.242$

and $\nu_F = 190\,\text{MHz}$

So mode spacing is $236\,\text{MHz}$

b) Get: