Phys 532 Assignment 4 Solutions

1. a) Spontaneous emission rate is $\frac{1}{t_s} = \frac{1}{3\,ms}$

b) $W = \frac{I\sigma}{h\nu} = \frac{1}{t_s}$

$$\boxed{I = \frac{h\nu}{\sigma t_s}}$$

Stimulated rate is $P_{st} = N \frac{c}{V} \sigma(\nu)$

Have $\sigma(\nu_0) = \frac{\lambda^2}{8\pi t_{sp}} g(\nu_0)$

$$g(\nu_0) = \frac{\Delta\nu/2\pi}{(\Delta\nu/2)^2} = \frac{2}{\pi\Delta\nu}$$

$$\sigma(\nu_0) = \frac{\lambda^2}{4\pi^2 t_{sp}\,\Delta\nu}$$

$$= \frac{(0.7\,\mu m)^2}{4\pi^2 \times 3\,ms \times 50\,GHz} = 8.28\times10^{-19}\,cm^2$$

So, need $N = \frac{1}{t_s}\frac{V}{c\sigma} = \frac{1}{3\,ms}\,\frac{100\,cm^3}{3\times10^{10}\,cm/s \cdot 8.3\times10^{-19}\,cm^2}$

$$\boxed{N = 1.34\times10^{12}\text{ photons}}$$

2. a) Have $I(\vec r) = I_0 \cos^2 kz \,\frac{w_0^2}{w(z)^2}\, e^{-2\rho^2/w(z)^2}$

$$w(z)^2 = w_0^2\left(1+\frac{z^2}{z_0^2}\right)$$

So $\int I(\vec r)\,dV = \int_{-d/2}^{d/2}\int_0^\infty I(\vec r)\, 2\pi\rho\, d\rho\, dz$

$$= \frac{\pi}{4} I_0\, w_0^2 d$$

from last week problem 3.5

So,
$$\rho_{sp}(\vec{r}=0) = \frac{I_0}{\frac{\pi}{4} I_0 w_0^2 d} \cdot c\,\sigma(\nu)$$

$$= \frac{4c}{\pi w_0^2 d}\,\sigma(\nu)$$

This is rate to emit into mode with frequency ν

Sum over longitudinal modes

$$\rho_{sp}' = \int \frac{4c}{\pi w_0^2 d}\,\sigma(\nu)\, M'(\nu)\, d\nu$$

$$M'(\nu) = \frac{\#\text{ of modes}}{\text{freq. range}}$$

TEM_{00} modes spaced by $\nu_F = \frac{c}{2d}$ so

$$M'(\nu) = \frac{2}{\nu_F} = \frac{4d}{c}$$

(two polarizations per mode)

$$\rho_{sp}' = \frac{16}{\pi}\,\frac{1}{w_0^2}\int \sigma(\nu)\, d\nu$$

$$\boxed{\rho_{sp}' = \frac{16}{\pi}\,\frac{1}{w_0^2}\, S}$$

Have total $\rho_{sp} = \frac{8\pi}{\lambda^2} S$, so $\boxed{\frac{\rho_{sp}'}{\rho_{sp}} = \frac{2}{\pi^2}\,\frac{\lambda^2}{w_0^2}}$

3. Have

$$\bar{g}(\nu) = \int_{-\infty}^{\infty} g(\nu - \nu') \, p(\nu') \, d\nu'$$

Where $g(\nu - \nu')$ is line shape function centered at $\nu_0 + \nu'$

$p(\nu')$ = prob that atom has center at $\nu_0 + \nu'$

Here ν' depends on z: $\nu' = \mu\beta z$

So $p(\nu')\,d\nu' = p(z)\,dz$

If all z's equally likely, then $p(z) = \frac{1}{L}$ $\left(-\frac{L}{2} < z < \frac{L}{2}\right)$

Have $\frac{d\nu'}{dz} = \mu\beta$, so

$$p(\nu') = \frac{1}{\mu\beta L}$$

$$\bar{g}(\nu) = \int_{-\mu\beta\frac{L}{2}}^{\mu\beta\frac{L}{2}} g(\nu - \nu') \frac{d\nu'}{\mu\beta L}$$

If $\Delta\nu \ll \mu\beta L$, then if $|\nu - \nu_0| > \frac{\mu\beta L}{2}$,

peak of g is outside range of integration: $\bar{g}(\nu) \approx 0$

If $|\nu - \nu_0| < \frac{\mu \beta L}{2}$, integrate over peak of g:

$$\bar{g}(\nu) \approx \frac{1}{\mu \beta L} \times 1$$

So, get

$$\boxed{\begin{aligned} \bar{g}(\nu) &= \frac{1}{\mu \beta L} & |\nu - \nu_0| < \frac{\mu \beta L}{2} \\ &= 0 & |\nu - \nu_0| > \frac{\mu \beta L}{2} \end{aligned}}$$

4. Line broadening:

radiative $\Delta \nu = \frac{1}{2\pi t_s} = 10 \text{ MHz}$

collision $\Delta \nu = \frac{f_{col}}{\pi} = 320 \text{ Hz}$

Doppler $\Delta \nu = 2.35 \frac{1}{\lambda} \sqrt{\frac{kT}{M}} = 2.35 \frac{1}{590\text{nm}} \sqrt{\frac{1.38 \times 10^{-23} \frac{J}{K} \cdot 400K}{23 \times 1.67 \times 10^{-27} \text{kg}}}$

$= 1.5 \text{ GHz}$

Doppler broadening is dominant

Absorption $\alpha(\nu) = N \frac{\lambda^2}{8\pi t_{sp}} \frac{g_2}{g_1} \bar{g}(\nu)$

On resonance, $\bar{g}(\nu_0) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{M}{kT}} = 6.2 \times 10^{-10} \text{ s}$

So, for $P_{1/2}$ state $g_2 = g_1$

$$\alpha = 3\times10^{17}\,m^{-3} \times \frac{(589.6\,nm)^2}{8\pi \cdot 16\,ns} \cdot 6.2\times10^{-10}\,s$$

$$= 161\,m^{-1} = \boxed{1.6\,cm^{-1}}$$

For the $P_{3/2}$ state, $g_2 = 2g_1$, so $\boxed{\alpha = 3.2\,cm^{-1}}$

5. a) Have $R = \frac{4\pi^2}{h} H_{12}^2\, \delta(E_2 - E_1 - h\nu)$

Compare to $\rho_{sp} = \frac{c}{V}\sigma(\nu)$

* Could include W_{stim}, but then Hamiltonian also needs to include incident field.

$$\sigma(\nu) = \frac{V}{c} R$$

$$= \frac{4\pi^2 V}{hc} H_{12}^2\, \delta(E_2 - E_1 - h\nu)$$

If $\nu_0 = \frac{E_2 - E_1}{h}$

$$\boxed{\sigma(\nu) = \frac{4\pi^2 V}{h^2 c} H_{12}^2\, \delta(\nu_0 - \nu)}$$

b) Get $\sigma = \frac{4\pi^2 V}{h^2 c} e^2 \frac{h\nu}{2V\varepsilon_0} |\hat{\varepsilon}\cdot\vec{d}_{12}|^2\, \delta(\nu_0 - \nu)$

$$= 4\pi^2 \nu \frac{e^2}{2\varepsilon_0 hc} |\hat{\varepsilon}\cdot\vec{d}_{12}|^2\, \delta(\nu_0 - \nu)$$

$$\boxed{\sigma = 4\pi^2 \alpha \nu\, |\hat{\varepsilon}\cdot\vec{d}_{12}|^2\, \delta(\nu_0 - \nu)}$$