Phys 532 Assigment 5 Solutions

1. Set up rate equations:

$$\frac{dN_3}{dt} = +R - \frac{1}{\tau_3} N_3 = 0 \quad \Rightarrow \quad N_3 = R\tau_3$$

$$\frac{dN_2}{dt} = +\frac{1}{\tau_{32}} N_3 - \frac{1}{\tau_2} N_2 = 0 \quad \Rightarrow N_2 = \tau_2 \frac{\tau_3}{\tau_{32}} R$$

$$\frac{dN_1}{dt} = +\frac{1}{\tau_{31}} N_3 + \frac{1}{\tau_{21}} N_2 - \frac{1}{\tau_1} N_1 = 0$$

$$N_1 = \frac{\tau_1}{\tau_{31}} N_3 + \frac{\tau_1}{\tau_{21}} N_2$$

$$N_1 = \tau_1 R \left(\frac{\tau_3}{\tau_{31}} + \frac{\tau_2}{\tau_{21}} \frac{\tau_3}{\tau_{32}} \right)$$

So $\Delta N_0 = N_2 - \frac{g_2}{g_1} N_1$

$$= R\left[\tau_2 \frac{\tau_3}{\tau_{32}} - \frac{g_2}{g_1} \left(\frac{\tau_1 \tau_3}{\tau_{31}} + \frac{\tau_1 \tau_2 \tau_3}{\tau_{21} \tau_{32}} \right) \right]$$

$$\boxed{\Delta N_0 = \tau_3 R \left[\frac{\tau_2}{\tau_{32}} - \frac{g_2}{g_1} \tau_1 \left(\frac{1}{\tau_{31}} + \frac{\tau_2}{\tau_{21} \tau_{32}} \right) \right]}$$

2. Total gain $G = e^{\gamma d}$

$d = 1$ cm

$G = 2.5$

So want $\gamma = 0.91 \text{ cm}^{-1}$

Have $\gamma = \frac{\lambda^2}{8\pi t_s} g(\nu) \Delta N_0$

For $2 \to 1$ transition, $t_s = \tau_{21} = 1\,ms$

use $g(\nu) \approx \frac{2}{\pi \Delta\nu_{12}} = 6.4\times10^{-12}\,s$

$\lambda = 800\,nm / n = 471\,nm$

($\Rightarrow \Delta N = 1.6\times10^{18}\,cm^{-3}$)

[Compare $N_a = 10^{19}\,cm^{-3}$

$\frac{\Delta N}{N_a} = 0.16$, some depletion of state 0... will change ΔN formula]

And $\frac{1}{\tau_3} = \frac{1}{\tau_{30}} + \frac{1}{\tau_{32}} = \frac{1}{50\,ns}$

$\frac{1}{\tau_2} = \frac{1}{1\,ms} + \frac{1}{1\,ms} = \frac{1}{500\,\mu s}$

So $$\Delta N = R \times 50\,ns\left[\frac{500\,\mu s}{100\,ns} - \frac{4}{2}\cdot 25\,ns\left(\frac{1}{2\,ms} + \frac{500\,\mu s}{1\,ms \cdot 100\,ns}\right)\right]$$

$$= R \times 50\,ns\left[5000 - 0.25\right]$$

$$\simeq 250\,\mu s \times R \equiv \tau' R$$

So, require $$0.91\,cm^{-1} = \frac{\lambda^2}{8\pi t_s}\,g(\nu)\,R\,\tau'$$

$$R = \frac{91\,m^{-1} \cdot 8\pi \cdot 1\,ms \cdot 1.6}{(471\,nm)^2\,(250\,\mu s)\,(6.4\times10^{-12}\,s)}$$

$$= 6.4\times10^{27}\,\frac{1}{m^3\cdot s}$$

But pump rate R is given by

$$R = W_{0\to3}\,N_a \qquad (W_{0\to3} = 644\,s^{-1})$$

$$= \frac{\lambda^2}{8\pi t_s}\,g(\nu)\,\frac{I}{h\nu}\,\frac{g_3}{g_0}\,N_a$$

for $\lambda = 500\,nm / n = 294\,nm$

$t_s = 100\,ns$

$h\nu = \frac{hc}{\lambda} = 3.98 \times 10^{-19}\,J$

$g(\nu) \approx \frac{2}{\pi \Delta\nu_{02}} = 1.3 \times 10^{-14}\,s$

$N_a = 10^{25}\,m^{-3}$

So require

$$I_p = \frac{6.4 \times 10^{27}\,m^{-3}s^{-1}}{10^{25}\,m^{-3}} \cdot \frac{8\pi(100\,ns)}{(294\,nm)^2} \cdot \frac{1}{1.3\times10^{-14}\,s} \cdot \frac{2}{8} \cdot 3.98 \times 10^{-19}\,J$$

$$\boxed{I_p = 1.4 \times 10^5 \frac{W}{m^2}}$$

3. Set up rate equations with 1-2 transition being driven.

Define $W_{1\to2} = Wg_2$

$W_{2\to1} = Wg_1$

So $W = \sigma(\nu) \frac{I}{h\nu}$

Then

$$\frac{dN_3}{dt} = +R - \frac{1}{\tau_3} N_3 = 0 \quad \Rightarrow N_3 = \tau_3 R$$

$$\frac{dN_2}{dt} = + \frac{1}{\tau_{32}} N_3 - \frac{1}{\tau_2} N_2 - Wg_1 N_2 + Wg_2 N_1 = 0$$

$$\frac{dN_1}{dt} = + \frac{1}{\tau_2} N_2 - \frac{1}{\tau_1} N_1 + Wg_1 N_2 - Wg_2 N_1 = 0$$

Add equations for $\frac{dN_2}{dt}$ & $\frac{dN_1}{dt}$:

$$\frac{1}{\tau_{32}} N_3 - \frac{1}{\tau_1} N_1 = 0$$

$$N_1 = \frac{\tau_1}{\tau_{32}} N_3 = \tau_1 \frac{\tau_3}{\tau_{32}} R$$

Then have

$$N_2\left(\frac{1}{\tau_2} + W g_1\right) = \frac{1}{\tau_{32}} N_3 + W g_2 N_1$$

$$= \frac{\tau_3}{\tau_{32}} R + W g_2 \tau_1 \frac{\tau_3}{\tau_{32}} R$$

$$N_2 = \frac{\tau_2 \tau_3}{\tau_{32}} \frac{1 + W g_2 \tau_1}{1 + W g_1 \tau_2} R$$

Gives

$$\Delta N = N_2 - \frac{g_2}{g_1} N_1$$

$$= \frac{\tau_3}{\tau_{32}} \left[\frac{\tau_2 + W g_2 \tau_2}{1 + W g_1 \tau_2} - \frac{g_2}{g_1} \tau_1 \right] R$$

$$= \frac{\tau_3}{\tau_{32}} \left[\frac{\tau_2 + W g_2 \tau_2 - \frac{g_2}{g_1} \tau_1 - W g_2 \tau_1 \tau_2}{1 + W g_1 \tau_2} \right] R$$

$$\Delta N = \frac{\tau_3}{\tau_{32}} \left[\frac{\tau_2 - \frac{g_2}{g_1} \tau_1}{1 + W g_1 \tau_2} \right]$$

Note that from problem 1, if $\tau_{20} = \tau_{31} = \infty$,

then $\tau_2 = \tau_{21}$, and get

$$\Delta N_0 = \tau_3 R \left[\frac{\tau_2}{\tau_{32}} - \frac{g_2}{g_1} \frac{\tau_1}{\tau_{32}} \right]$$

So have
$$\Delta N = \frac{\Delta N_0}{1 + W g_1 \tau_2}$$

So
$$\frac{I}{I_s} = W g_1 \tau_2$$

$$\text{but } W g_1 = W_{2\to 1} = \frac{\lambda^2}{8\pi t_s} g(\nu) \frac{I}{h\nu}$$

$$\frac{1}{I_s} = \frac{\lambda^2}{8\pi t_s} g(\nu) \frac{1}{h\nu} \tau_2 \qquad \tau_2 = t_s \text{ here}$$

$$I_s = \frac{8\pi h\nu}{\lambda^2 g(\nu)}$$

4. Go through rate equations for 2 level system.

Again, take
$$W_{1\to 2} = W g_2$$
$$W_{2\to 1} = W g_1$$

Then
$$\frac{dN_2}{dt} = -\frac{1}{\tau_2} N_2 + W g_2 N_1 - W g_1 N_2 = 0$$

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt}$$

$$N_2\left(\frac{1}{\tau_2} + Wg_1\right) = Wg_2 N_1$$

$$N_2 = N_1 \frac{Wg_2\tau_2}{1+Wg_1\tau_2}$$

If $N_a = N_1 + N_2$, get

$$N_a = N_1\left(1 + \frac{Wg_2\tau_2}{1+Wg_1\tau_2}\right)$$

$$N_1 = N_a \frac{1}{1+\frac{Wg_2\tau_2}{1+Wg_1\tau_2}} = N_a \frac{1+Wg_1\tau_2}{1+W(g_1+g_2)\tau_2}$$

and $$N_2 = N_a \frac{Wg_2\tau_2}{1+W(g_1+g_2)\tau_2}$$

So $$\Delta N = N_2 - \frac{g_2}{g_1}N_1$$

$$= N_a\left[\frac{Wg_2\tau_2}{1+W(g_1+g_2)\tau_2} - \frac{g_2}{g_1}\frac{(1+Wg_1\tau_2)}{1+W(g_1+g_2)\tau_2}\right]$$

$$= N_a\left[\frac{-g_2/g_1}{1+W(g_1+g_2)\tau_2}\right]$$

$\Delta N < 0$, make sense, should get absorbtion.

Since $\alpha = \frac{\lambda^2}{8\pi t_s} g(\nu)\,(-\Delta N)$

then have

$$\frac{I}{I_s} = W(g_1+g_2)\,\tau_2$$

Recall $W_{2\to1} = \frac{\lambda^2}{8\pi t_s} g(\nu)\frac{I}{h\nu} = W g_1$

so $\frac{1}{I_s} = \frac{\lambda^2 \tau_2}{8\pi t_s}\,\frac{g_1+g_2}{g_1}\,g(\nu)\,\frac{1}{h\nu}$

But here $\tau_2 = t_s$, so

$$I_s = \frac{8\pi}{\lambda^2}\,\frac{g_1}{g_1+g_2}\,\frac{h\nu}{g(\nu)}$$

Now for $g(\nu)$, need to use homogeneous line shape, so at peak $g(\nu_0) = \frac{2}{\pi\Delta\nu}$

$\Delta\nu = \frac{1}{2\pi t_s}$

$g(\nu_0) = 4t_s$

and $$\boxed{I_s = \frac{2\pi}{\lambda^2}\,\frac{g_1}{g_1+g_2}\,\frac{h\nu}{t_s}}$$

So, for the $3p_{3/2}$ transition,

$$I_s = \frac{2\pi}{(589\,nm)^3}\,\frac{2}{2+4}\,\frac{hc}{16\,ns}$$

$$\boxed{I_s = 127\ W/m^2}$$

For the $3p_{1/2}$ transition, $I_s = \frac{2\pi}{(589\,nm)^3}\,\frac{2}{2+2}\,\frac{hc}{16\,ns}$

$$\boxed{I_s = 191\ W/m^2}$$

5. Here frequency shift $\nu_z = \mu \beta z$

with probability distribution

$$p(\nu_z) = \frac{1}{\mu\beta L} \qquad |\nu_z| < \frac{\mu\beta L}{2}$$
$$= 0 \qquad |\nu_z| > \frac{\mu\beta L}{2}$$

For two-level system, have

$$I_s = \frac{2\pi}{\lambda^2} \frac{g_1}{g_1+g_2} \frac{h\nu}{t_s}$$
$$= \frac{\pi}{\lambda^2} \frac{h\nu}{t_s} \qquad \text{for equal degeneracies}$$

So set $$\gamma_z(\nu) = \frac{b\,\Delta\nu/2\pi}{(\nu-\nu_z-\nu_0)^2 + (\Delta\nu_s/2)^2}$$

for $\Delta\nu_s = \Delta\nu\left[1 + \frac{I}{I_s(\nu_0)}\right]^{1/2}$

Then $$\bar{\gamma}(\nu) = \int_{-\infty}^{\infty} \gamma_z(\nu)\, p(\nu_z)\, d\nu_z$$

$$= \int_{-\frac{\mu\beta L}{2}}^{+\frac{\mu\beta L}{2}} \frac{b\,\Delta\nu/2\pi}{(\nu-\nu_z-\nu_0)^2 + (\Delta\nu_s/2)^2} \frac{1}{\mu\beta L}\, d\nu_z$$

If $\Delta\nu \ll \frac{\mu\beta L}{2}$ and I is not much larger than I_s, then $\Delta\nu_s \ll \frac{\mu\beta L}{2}$ as well.

Then as long as $|\nu - \nu_0| < \frac{\mu\beta L}{2}$, can extend limits of integrand to infinity

So $$\bar{\gamma}(\nu) \approx b \frac{\Delta\nu}{\Delta\nu_s} \frac{1}{\mu\beta L} \int_{-\infty}^{\infty} \frac{\Delta\nu_s/2\pi}{(\nu-\nu_z-\nu_0)^2+(\Delta\nu_s/2)^2} \, d\nu_z$$

$$= b \frac{\Delta\nu}{\Delta\nu_s} \frac{1}{\mu\beta L}$$

$$= \frac{b}{\mu\beta L} \frac{1}{\sqrt{1+I/I_s}}$$

where $b = -N_a \frac{\lambda^2}{8\pi t_s}$ (Since $\Delta N_0 = -N_a$ with no pumping)

or, $$\boxed{\bar{\alpha}(\nu) = \frac{\lambda^2}{8\pi t_s} \bar{g}(\nu) \frac{N_a}{\sqrt{1+I/I_s}}}$$

where $$\bar{g}(\nu) = \begin{cases} \frac{1}{\mu\beta L} & |\nu-\nu_0| < \frac{\mu\beta L}{2} \\ 0 & |\nu-\nu_0| > \frac{\mu\beta L}{2} \end{cases}$$

$$I_s = \frac{\pi}{\lambda^2} \frac{h\nu}{t_s}$$