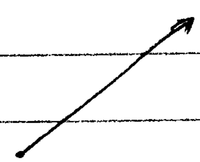
Impermeability tensor $\overleftrightarrow{\eta} = \epsilon_0 (\overleftrightarrow{\epsilon})^{-1}$

In principal coordinate system of crystal (x, y, z) [or $(1,2,3)$] $\overleftrightarrow{\eta}$ is diagonal with principal values

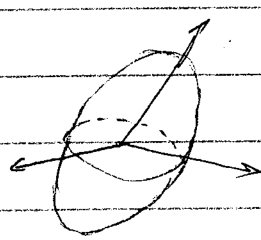
$$\frac{\epsilon_0}{\epsilon_x} = \frac{1}{n_x^2} \qquad \frac{\epsilon_0}{\epsilon_y} = \frac{1}{n_y^2} \qquad \frac{\epsilon_0}{\epsilon_z} = \frac{1}{n_z^2}$$

(or equivalently: $\frac{\epsilon_0}{\epsilon_1} = \frac{1}{n_1^2}$, $\frac{\epsilon_0}{\epsilon_2} = \frac{1}{n_2^2}$, $\frac{\epsilon_0}{\epsilon_3} = \frac{1}{n_3^2}$.)

Geometrical representation of vectors and tensors



vector $\vec{A}$: arrow with amplitude and direction



A useful representation of a symmetrical second-rank tensor ($\overleftrightarrow{\epsilon}$, for example) is a quadratic surface (an ellipsoid) defined by

$$\sum_{ij} \epsilon_{ij} x_i y_i = 1$$

It's a 3D surface.

Index ellipsoid: quadratic representation of the electric impermeability tensor $\vec{\vec{\eta}}$, 3D surface

$$\sum_{i,j} = \eta_{ij} x_i x_j = 1$$

in "lab" coordinates (x_1, x_2, x_3)

typically, x_3 = direction of propagation of light

usually written in "piezoelectric" convention

$$\eta_1 x_1^2 + \eta_2 x_2^2 + \eta_3 x_3^2 + 2\eta_4 x_2 x_3 + 2\eta_5 x_1 x_3 + 2\eta_6 x_1 x_2 = 1$$

where $\eta_1 \equiv \eta_{11}$ $\eta_2 \equiv \eta_{22}$ $\eta_3 \equiv \eta_{33}$

$\eta_4 \equiv \eta_{23} \equiv \eta_{32}$ $\eta_5 \equiv \eta_{13} \equiv \eta_{31}$ $\eta_6 \equiv \eta_{12} \equiv \eta_{21}$.

Note: crystal principal axes = axes of ellipsoid

in crystal principal coordinates (x, y, z)

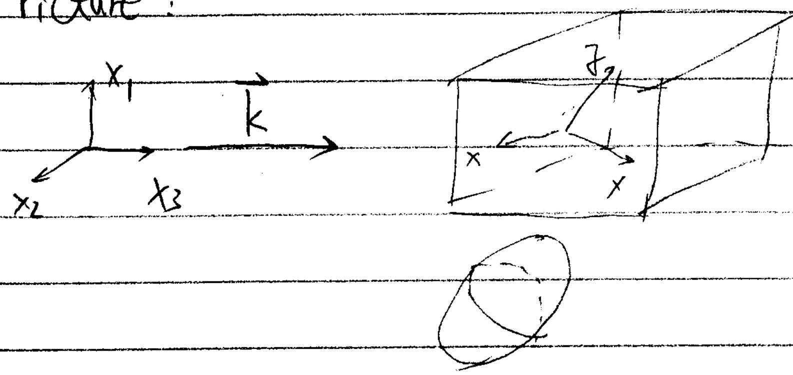
$$\eta_1 \to \frac{1}{n_x^2} \quad \eta_2 \to \frac{1}{n_y^2} \quad \eta_3 \to \frac{1}{n_z^2}$$

$$\eta_4 \to 0 \quad \eta_5 \to 0 \quad \eta_6 \to 0$$

So $$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

or $$\frac{x^2 + y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$ for uniaxial crystal

Picture:

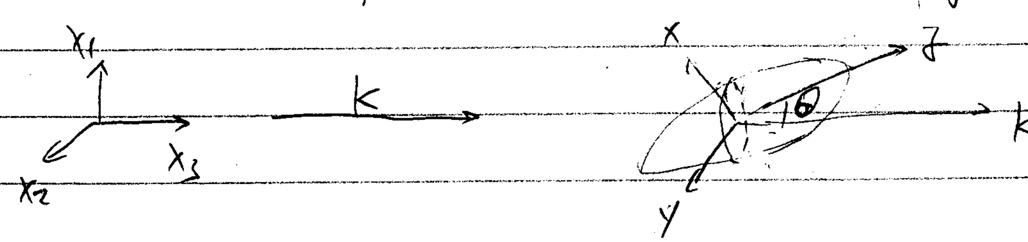


Now $\vec{E} \perp \vec{k}$ for light, so polarization vector $\hat{\varepsilon}$ lies in $x_1 - x_2$ plane

Intersection of plane of polarization with index ellipsoid yield an ellipse.

equation: $\eta_1 x_1^2 + \eta_2 x_2^2 + 2\eta_6 x_1 x_2 = 1$

For uniaxial crystal, this can be seen geometrically



Set $(\hat{y} \rightarrow \hat{x}_2)$, rotate around $\hat{y}$ by θ

project to x_1 x_2 plance

$x \rightarrow x_1 \cos\theta$

$z \rightarrow x_1 \sin\theta$

$y \rightarrow x_2$.

So $$\frac{x^2+y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

$$\rightarrow \frac{x_1^2\cos^2\theta}{n_o^2} + \frac{x_2^2}{n_o^2} + \frac{x_1^2\sin^2\theta}{n_e^2} = 1$$

"Index ellipse"

Then major axes of ellipse give effective indices of refraction

$$\frac{1}{n_1^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

$$\frac{1}{n_2^2} = \frac{1}{n_o^2}.$$

Always get two effective indices (book uses n_a, n_b)

Example: KDP with axis z at $30°$ to $\vec{k}$ ($\hat{x}_3$)

Pick $\hat{x}_1$ in $\hat{z}\,\hat{x}_3$ plane
$\hat{x}_2 = \hat{y}$.

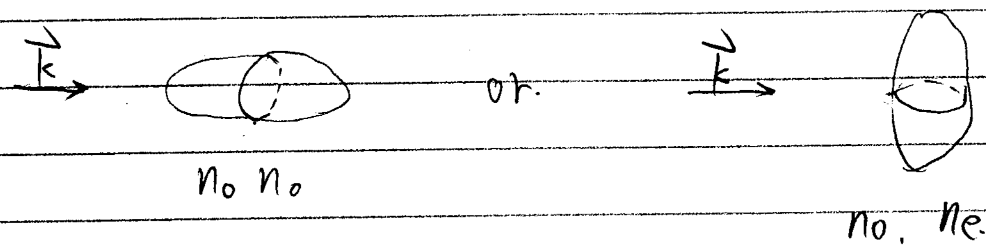
Light polarized along $\hat{x}_1$ has

$$\frac{1}{n_1^2} = \frac{\cos^2 30°}{n_o^2} + \frac{\sin^2 30°}{n_e^2}$$

$$= \frac{3}{4n_o^2} + \frac{1}{4n_e^2}$$

$n_o = 1.47$, $n_e = 1.51$ $\rightarrow$ $n_1 = 1.50$

So for given n_e & n_o, can adjust
→ one is always n_o
→ other ranges from n_o to n_e depending on angle.



Electro-optic effect

Best described in term of impermeability tensor.

$$\eta_{ij}(\vec{E}) = \eta_{ij} + \sum_k r_{ijk} E_k + \sum_{lm} S_{ijlm} E_l E_m$$

↑ Pockels ↑ kerr.

Labeling convention.
usually use tensor equation to describe effect, but label r, s with pieze-electric convention

$r_{xyx} = r_{61}$: change in η_{xy} due to field E_x

$S_{zzxx} = S_{31}$: change in η_{zz} due to term E_x^2.

Write r as 3×6 matrix
s as 6×6 matrix

To work with them, translate back to tensor form.

Depending on crystal symmetry, many elements of r & s are zero.

S&T table 18.2-2, 18.2-3 gives a few examples

Example: KDP (symmetry class $\bar{4}2m$) has

$$r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_{41} & 0 & 0 \\ 0 & r_{41} & 0 \\ 0 & 0 & r_{63} \end{bmatrix}$$

r_{yzx} $\quad$ r_{xzy}

$r_{41} = r_{52} = 8.6$ pm/V

$r_{63} = 10.6$ pm/V

r_{xyz}.

Then index ellipsoid becomes

$$\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} + 2 r_{41} yz E_x + 2 r_{41} xz E_y + 2 r_{63} xy E_z = 1$$

$(n_o = 1.51, \ n_e = 1.47)$

How can we use this to make a modulator.