04/22/05 Lecture 34.

So we have notation for complex amplitudes

$$P_i(\omega_3) = \sum_{jk} 2d'_{ijk}\, \underbrace{E_j(\omega_1)\, E_k(\omega_2)}$$

Polarization amp at $\omega_3 = \omega_1 + \omega_2$

Include factor of $\frac{1}{2}$ if $\omega_1 = \omega_2$

Incident field components at ω_1, ω_2

(P_i) is source for wave equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

Component $P_i(\omega_3)$ excites field at ω_3

Say $\vec{E}(\vec{r},t) = \vec{E}(\omega_1)\, e^{i(\omega_1 t - \vec{k}_1 \cdot \vec{r})}$
$+ \vec{E}(\omega_2)\, e^{i(\omega_2 t - \vec{k}_2 \cdot \vec{r})} + \vec{E}(\omega_3)\, e^{i(\omega_3 t - \vec{k}_3 \cdot \vec{r})}$

Get long interaction lengths with collinear propagation

Say waves propagate along z.

Then field at ω_3 has $\vec{E}(\omega_3, z)\, e^{i(\omega_3 t - k_3 z)}$
$= \vec{E}_3\, e^{i(\omega_3 t - k_3 z)}$

$$\nabla^2 \vec{E} = \frac{\partial^2}{\partial z^2}\vec{E} = \Big(\underbrace{\cancel{\frac{\partial^2 \vec{E}}{\partial z^2}}}_{\text{neglect}} - 2ik_3 \frac{\partial \vec{E}(\omega_3)}{\partial z} - k_3^2\, \vec{E}(\omega_3)\Big)\, e^{i(\omega_3 t - k_3 z)}$$

Wave equation (for ω_3) becomes

$$\left[-2ik_3\frac{\partial E_i(\omega_3)}{\partial z} - k_3^2 E_i(\omega_3) + k_3^2 E_i(\omega_3)\right] e^{i(\omega_3 t - k_3 z)}$$

$$= -2\mu_0\omega_3^2 \sum_{jk} d'_{ijk} E_j(\omega_1) E_k(\omega_2) e^{i(\omega_3 t - \vec{k}_1\cdot\vec{r} - \vec{k}_2\cdot\vec{r})}$$

$$\frac{\partial E_i(\omega_3)}{\partial z} = -i\mu_0 \frac{\omega_3^2}{k_3} \sum_{jk} d'_{ijk} E_j(\omega_1) E_k(\omega_2) e^{i(\vec{k}_3 - \vec{k}_2 - \vec{k}_1)\cdot\vec{r}}$$

$$k_3 = \frac{\omega_3}{c} = n_3\,\omega_3\sqrt{\epsilon_0\mu_0}$$

$$= -i\frac{\omega_3}{n_3}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{jk} d'_{ijk} E_j(\omega_1) E_k(\omega_2) e^{i\Delta\vec{k}\cdot\vec{r}}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega = \eta_0$$

Units: $\frac{V}{m^2} = \frac{1}{s}\cdot\Omega\cdot\frac{C}{V^2}\cdot\left(\frac{V}{m}\right)^2$

$$= \frac{1}{m^2}\cdot\Omega\cdot\frac{C}{s} = \frac{1}{m^2}\cdot\Omega\cdot A = \frac{V}{m^2} \;\checkmark$$

Similarly

$$\frac{\partial E_i(\omega_1)}{\partial z} = -i\frac{\omega_1}{n_1}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{jk} d'_{ijk} E_j(\omega_3) E_k^*(\omega_2) e^{i(\vec{k}_1 + \vec{k}_2 - \vec{k}_3)\cdot\vec{r}}$$

$$\frac{\partial E_i(\omega_2)}{\partial z} = -i\frac{\omega_2}{n_2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sum_{jk} d'_{ijk} E_j^*(\omega_1) E_k(\omega_3) e^{i(\vec{k}_2 + \vec{k}_1 - \vec{k}_3)\cdot\vec{r}}$$

Assume that $E(\omega_3)$ is always small

Then $\frac{\partial E(\omega_1)}{\partial z}$ + $\frac{\partial E(\omega_2)}{\partial z}$ are negligible.

Then can solve for $E_i(\omega_3)$ propagate through distance L

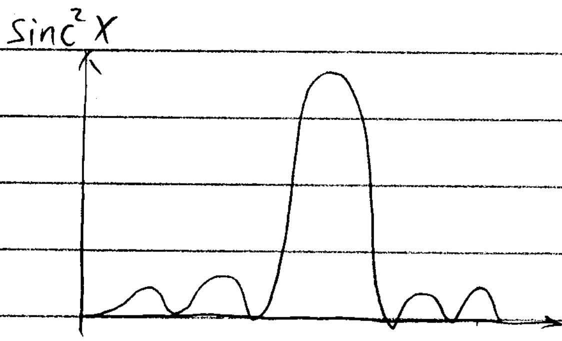
$$E_i(\omega_3) = -i\frac{\omega_3}{n_3}\eta_0 \sum_{jk} d'_{ijk}\, E_j(\omega_1)\, E_k(\omega_2) \underbrace{\int_0^L e^{i\Delta k z}\, dz}$$

$$= \frac{1}{i\Delta k}\left(e^{i\Delta k L} - 1\right)$$

$$= \frac{1}{i\Delta k} e^{i\frac{\Delta k L}{2}}\left(e^{i\frac{\Delta k L}{2}} - e^{-i\frac{\Delta k L}{2}}\right)$$

$$= L e^{i\frac{\Delta k L}{2}}\left[\frac{\sin\left(\frac{\Delta k L}{2}\right)}{\frac{\Delta k L}{2}}\right]$$

$\text{sinc}\, x = \frac{\sin x}{x}$ function, like single slit diffraction pattern from optics



$\text{sinc}\, x = 0$

when $x = \frac{\Delta k L}{2} = \pi$

$L = 2\pi / \Delta k$

$$\frac{1}{L} = \frac{1}{\lambda_3} - \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$$

so if we want to use $L \gg \lambda$, need $\Delta k = 0$

if $\Delta k = 0$

$$\boxed{E_i(\omega_3) = -i\frac{\omega_3}{n_3}\eta_0 L \sum_{jk} d'_{ijk}\, E_j(\omega_1)\, E_k(\omega_2)}$$

$$\Delta k = k_3 - k_1 - k_2$$
$$= n_3 \frac{\omega_3}{c} - n_1 \frac{\omega_1}{c} - n_2 \frac{\omega_2}{c}$$

Simplify to the case of harmonic generation

$$\omega_1 = \omega_2 = \omega \,, \quad \omega_3 = 2\omega$$

$$\Delta k = \frac{2\omega}{c} \left[n(2\omega) - n(\omega) \right]$$

But in a crystal $n = n(\omega)$: dispersion

Typically n varies by few % over factor of 2 in ω

If we want $\Delta k \ll \frac{2\pi}{L}$

$$\frac{4\pi}{\lambda} \cdot \Delta n \ll \frac{2\pi}{L}$$

$$\Delta n \ll \frac{\lambda}{L}$$

if $\lambda = 1 \mu m$, $L = 5\,mm$

need $\Delta n \ll 10^{-4}$

So how can we do this ?
use birefringent crystal : phase-matching

If $\vec{E}(\omega) \perp \vec{E}(2\omega)$ then one has $n = n_o$
the other has $n = n_e$

Possible to get $n_o(\omega) = n_e(2\omega)$ or vice verse.

Examples, $LiNbO_3$ can be used to double
Nd:YAG laser 1064 nm → 532 nm

has

λ	n_o	n_e
1064 nm	2.2338	2.1539
532 nm	2.3310	2.2378

$$n_o(\omega) \simeq n_e(2\omega)$$

To get just right, can adjust temperature
find that n_o & n_e are soft functions of T

For $LiNbO_3$, get indices equal at $T = 60°C$

Of course, need correct d element:

Output is $E_z(2\omega)$, input is $E_x(\omega)$ and/or $E_y(\omega)$

So need d_{zxx}, d_{zyy}, or d_{zxy}
d_{31} d_{32} d_{36}

Have $d_{31} = d_{32} = 4 \times 10^{-23} \frac{C}{V^2}$ so it works

Get $\boxed{E_z(\omega_3) = -i \frac{\omega_3}{n} \eta_0 L \frac{d_{31}}{2} E_x(\omega_1)^2}$

Methods of phase-matching:

A. Find crystal with $\Delta k \approx 0$ for appropriate polarizations of beams

Fine adjust with temperature
"non-critical phase matching"

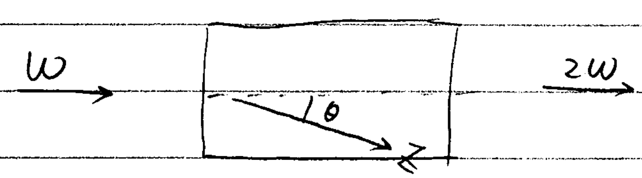
Not always possible / convenient

B. Use L small enough that Δn doesn't matter
Typically $L < 1mm$

make up for small L with large $E(\omega_1)$, $E(\omega_2)$
$\Rightarrow$ great for pulsed lasers

C. use birefringence, but adjust angle of crystal.
"critical PM"

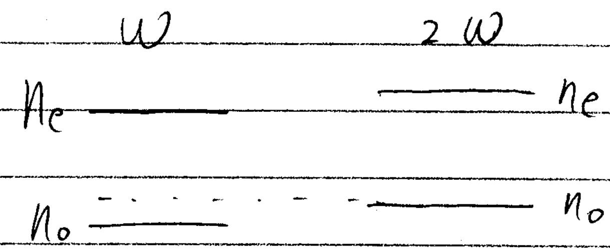
Consider SHG $\omega \rightarrow 2\omega$ in uniaxial crystal.



Have $n_a = n_o$ $\qquad \dfrac{1}{n_b^2} = \dfrac{\cos^2\theta}{n_o^2} + \dfrac{\sin^2\theta}{n_e^2}$

n_b ranges from n_o to n_e

Picture
($n_e > n_o$)



Want to adjust θ to make $n_b = n_a = n_o$

Here, means pick ω light to be e- polarized ↕
2ω light to be o- polarized ⊙

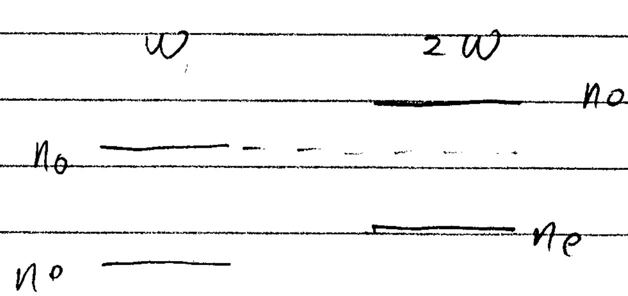
Then solve for θ:

$$\frac{1}{n_o(2\omega)^2} = \frac{\cos^2\theta}{n_o(\omega)^2} + \frac{\sin^2\theta}{n_e(\omega)^2}$$

$$= \frac{1}{n_o(\omega)^2} - \frac{\sin^2\theta}{n_o(\omega)^2} + \frac{\sin^2\theta}{n_e(\omega)^2}$$

$$\sin^2\theta = \frac{\frac{1}{n_o(\omega)^2} - \frac{1}{n_o(2\omega)^2}}{\frac{1}{n_o(\omega)^2} - \frac{1}{n_e(\omega)^2}}.$$

If $n_o > n_e$, reserved



use 2ω light e-polarized.

get $$\sin^2\theta = \frac{\frac{1}{n_o(\omega)^2} - \frac{1}{n_o(2\omega)^2}}{\frac{1}{n_e(2\omega)^2} - \frac{1}{n_o(2\omega)^2}}.$$