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01/31/05 Lecture 6.

Gaussian beam:

$$u(\vec{r}) = A_0 \frac{W_0}{W(z)} e^{-\frac{\rho^2}{W(z)^2}} e^{-ik\left[z + \frac{\rho^2}{2R(z)} - \zeta(z)\right]}$$

$$z_0 = \frac{\pi W_0^2}{\lambda}$$

$$W(z) = W_0 \sqrt{1 + \frac{z^2}{z_0^2}}$$

$$R(z) = \frac{z^2 + z_0^2}{z}$$

$$\zeta(z) = \tan^{-1} \frac{z}{z_0}$$

$$I(\vec{r}) = I_0 e^{-2\rho^2/W(z)^2}$$

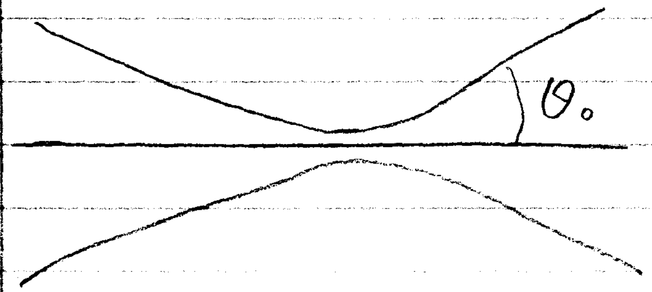
$$I_0 = \frac{2P}{\pi W^2}$$

One last property:

For $z \gg z_0$ $\quad W(z) \rightarrow W_0 \frac{z}{z_0}$

$$= \frac{\lambda}{\pi} \frac{z}{W_0}$$

beam size $W(z)$ proportional to z, so beam is expanding like a cone



Divergence angle $\theta_0 = \frac{W}{z} = \frac{\lambda}{\pi W_0}$

consistent with general rule for diffraction.

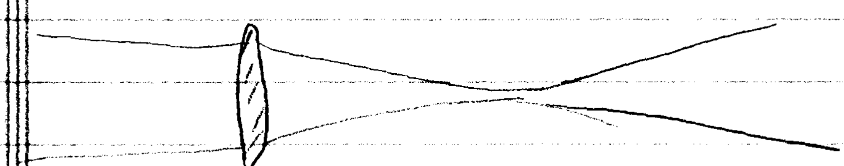
Light confined to region d, diffracts at angle $\theta \sim \frac{\lambda}{d}$.

Non-laser sources typically diverge much faster:

so laser beam is said to be "diffraction-limited"

Moreover, for convergence:

To achieve tighter spot, need to focus at steeper angle:



Note: paraxial approximation only valid for $\theta \lesssim 15°$ or so.

Propagation of Gaussian beam: through optical systems

Remarkable result:

If Gaussian beam with complex parameter $q_1 = z + iz_0$

passes through optical system with ray matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then beam output from system has parameter:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

(1) free propagation, distance d.

beam doesn't change, just distance from center.

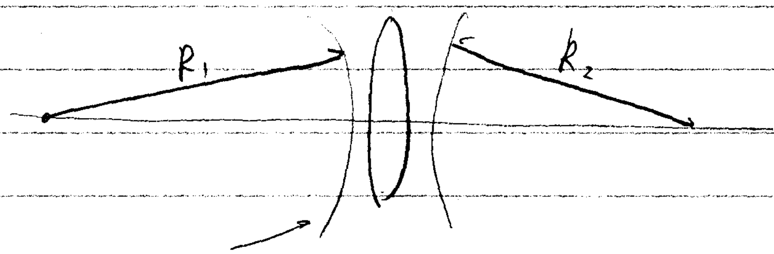
So if $q_1 = z + iz_0$, then $q_2 = (z+d) + iz_0$

Matrix is $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

so $q_2 = \dfrac{Aq_1 + B}{Cq_1 + D}$

(2) thin lens:

Don't change width W, but do change radius of curvature R



Looks like piece of spherical wave.

For actual spherical waves, know that:

$$\frac{1}{f} = \frac{1}{R_1} - \frac{1}{R_2} \qquad (R_2 < 0 \text{ for converging beam})$$

So $$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

But remember $$\frac{1}{q_1} = \frac{1}{R_1} - \frac{i\lambda}{\pi W^2}$$

So $$\frac{1}{q_2} = \frac{1}{R_2} - \frac{i\lambda}{\pi W^2} = \frac{1}{R_1} - \frac{1}{f} - \frac{i\lambda}{\pi W^2}$$

$$= \frac{1}{q_1} - \frac{1}{f}$$

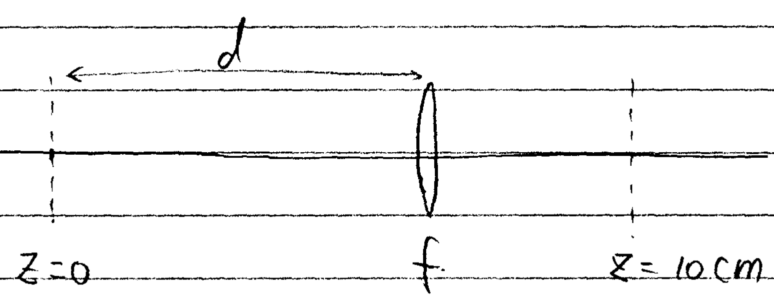
$$q_2 = \frac{1}{\frac{1}{q_1} - \frac{1}{f}} = \frac{q_1}{(-\frac{1}{f})q_1 + 1}$$

Matrix is $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$, so $q_2 = \dfrac{Aq_1 + B}{Cq_1 + D}$

Similar for other elementary matrices, and it also works for cascaded systems.

Example:

beam has $\lambda = 633\text{ nm}$, focussed at $z=0$, with waist $W_0 = 100\,\mu m$



want to refocus onto $z = 10\text{cm}$ plane, with new

$W_0' = 50\,\mu m$.

what lens focus length f & position d are required?

(6)

At input to lens, $q = d + i z_0$ $\qquad z_0 = \frac{\pi W_0^2}{\lambda} = 5$ cm.

after lens. $q' = \dfrac{q}{-\frac{1}{f}q + 1}$

$$= \frac{d + i z_0}{1 - \frac{d + i z_0}{f}}$$

$$= \frac{f(d + i z_0)}{f - d - i z_0}$$

We desire $\quad q' = d - 10\text{ cm} + i z_0'$

$$z_0' = \frac{\pi W_0'^2}{\lambda} = 1.25\text{ cm}$$

So $\quad d - 10 + i z_0' = \dfrac{f(d + i z_0)}{f - d - i z_0}$

$(d-10)(f-d) + z_0 z_0' + i\left[z_0'(f-d) - z_0(d-10)\right]$

$= f(d + i z_0)$

So $(d-10)(f-d) + z_0 z_0' = f d \qquad (1)$

$z_0'(f-d) - z_0(d-10) = f z_0 \qquad (2)$

Solve first:

$$fd - 10f - d^2 + 10d + z_0 z_0' = fd$$

$$f = \frac{1}{10}(z_0 z_0' + 10d - d^2)$$

Sub into second:

$$f(z_0' - z_0) - (z_0' + z_0)d + 10 z_0 = 0$$

$$\frac{z_0' - z_0}{10}(z_0 z_0' + 10d - d^2) - (z_0' + z_0)d + 10 z_0 = 0$$

Quadratic:

$$d^2 \frac{z_0 - z_0'}{10} - 2 z_0 d + \left[\frac{(z_0' - z_0) z_0 z_0'}{10} + 10 z_0\right] = 0$$

or $\quad 0.375 d^2 - 10d + 47.65 = 0$

$$d = \frac{1}{2 \times 0.375}\left[10 \pm \underbrace{\sqrt{10^2 - 4 \times 0.375 \times 47.65}}_{5.34}\right]$$

This is how laser beam works.

Apply to light in a cavity. (ch 9)

Before we have rays stably confined

How to have a confined Gaussian beam?

Cavity has matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Assume start with parameter q_1

after one round trip get

$$q_2 = \frac{A q_1 + B}{C q_1 + D}$$

want beam to return to same state

So require : $q_2 = q_1 = q$

Solve :

$$q = \frac{Aq + B}{Cq + D}$$

$$Cq^2 + Dq = Aq + B$$

$$Cq^2 + (D-A)q - B = 0$$

$$q = \frac{1}{2C}\left[A - D \pm \sqrt{(D-A)^2 + 4BC}\right]$$

use $\quad BC = AD - 1$

So $\quad (D-A)^2 + BC = D^2 - 2AD + A^2 + 4AD - 4$

$$= D^2 + 2AD + A^2 - 4$$

$$= (D+A)^2 - 4$$

So $$q = \frac{1}{C}\left[\frac{A-D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1}\right]$$

stability need $\quad \frac{A+D}{2} < 1$