Solutions for Examples

¢

1. Line broadening:

radiative: $\Delta\nu = \frac{1}{2\pi t_{sp}} = 10MHz$

collision: $\Delta\nu = \frac{f_{col}}{\pi} = 320\ Hz$

Doppler: $\Delta\bar{\nu} = 2.35\ \frac{1}{\lambda}\sqrt{\frac{kT}{M}}$

$$= 2.35\ \frac{1}{590\,nm}\sqrt{\frac{1.38\times10^{-23}\,J/k\cdot 400k}{23\times1.67\times10^{-27}\,kg}}$$

$$= 1.5\ GHz$$

Doppler broadening is dominant.

Absorption coefficient:

$$\alpha(\nu) = -\Delta N\cdot\frac{\lambda^2\bar{g}(\nu)}{8\pi t_{sp}}$$

$\Delta N = N_2 - \frac{g_2}{g_1}N_1$, here $N_2 \sim 0$, $N_1 = N$

so $\alpha(\nu) = N\frac{\lambda^2}{8\pi t_{sp}}\frac{g_2}{g_1}\bar{g}(\nu)$

On resonance

$$\bar{g}(\nu_0) = \frac{\lambda}{\sqrt{2\pi}}\sqrt{\frac{M}{kT}} \approx \frac{0.94}{\Delta\bar{\nu}} = 6.2\times10^{-10}\,s$$

So for $P_{1/2}$ state $g_2 = g_1$ ⑥

$$\alpha = 3 \times 10^{17}\,m^{-3} \times \frac{(589.6\,nm)^2}{8\pi \cdot 16\,nm} \times 6.2 \times 10^{-10}\,s$$

$$= 161\ m^{-1}$$

$$\boxed{= 1.6\ cm^{-1}}$$

For $P_{3/2}$ state, $g_2 = 2g_1$

So $\boxed{\alpha = 3.2\ cm^{-1}}$

2. Set up rate equations

$$\frac{dN_3}{dt} = +R - \frac{1}{\tau_3} N_3 = 0 \quad \Rightarrow \quad N_3 = R\tau_3$$

$$\frac{dN_2}{dt} = +\frac{1}{\tau_{32}} N_3 - \frac{1}{\tau_2} N_2 = 0 \quad \Longrightarrow \quad N_2 = \tau_2 \frac{\tau_3}{\tau_{32}} R$$

$$\frac{dN_1}{dt} = +\frac{1}{\tau_{31}} N_3 + \frac{1}{\tau_{21}} N_2 - \frac{1}{\tau_1} N_1 = 0$$

$$N_1 = \frac{\tau_1}{\tau_{31}} N_3 + \frac{\tau_1}{\tau_{21}} N_2$$

$$N_1 = \tau_1 R \left(\frac{\tau_3}{\tau_{31}} + \frac{\tau_2}{\tau_{21}} \cdot \frac{\tau_3}{\tau_{32}} \right)$$

So
$$\Delta N_0 = N_2 - \frac{g_2}{g_1} N_1$$

$$= R \left[\tau_2 \frac{\tau_3}{\tau_{32}} - \frac{g_2}{g_1} \left(\frac{\tau_1 \tau_3}{\tau_{31}} + \frac{\tau_1 \tau_2 \tau_3}{\tau_{21} \tau_{32}} \right) \right]$$

$$\Delta N_0 = \tau_3 R \left[\frac{\tau_2}{\tau_{32}} - \frac{g_2}{g_1} \tau_1 \left(\frac{1}{\tau_{31}} + \frac{\tau_2}{\tau_{21} \tau_{32}} \right) \right]$$

3. Total gain $G = e^{\gamma d}$ $\qquad d = 1\,cm$

$G = 2.5$

So want $\gamma = 0.91\ cm^{-1}$

Have
$$\gamma = \frac{\lambda^2}{8\pi t_{sp}} g(\nu) \Delta N_0$$

For $2 \to 1$ transition, $t_{sp} = \tau_{21} = 1 ms$

use $g(\nu) \approx \frac{2}{\pi \Delta \nu_{12}} = 6.4 \times 10^{-12} s$

$\lambda = 800 nm / n = 471 nm$

$(\Rightarrow \Delta N = 1.6 \times 10^{18} cm^{-3})$

[Note: compare $N_a = 10^{19} cm^{-3}$, $\frac{\Delta N}{N_a} = 0.16$
there is some depletion of state 0, this will actually change ΔN formula, but we don't consider that here]

And $\frac{1}{\tau_3} = \frac{1}{\tau_{30}} + \frac{1}{\tau_{32}} = \frac{1}{50 nm}$

$\frac{1}{\tau_2} = \frac{1}{1ms} + \frac{1}{1ms} = \frac{1}{500 \mu s}$

So $\Delta N = R \times 50ns \left[\frac{500 \mu s}{100 ns} - \frac{4}{2} \cdot 25ns \left(\frac{1}{2ms} + \frac{500 \mu s}{1ms \cdot 100ns} \right) \right]$

$= R \times 50ns [5000 - 0.25]$

$= 250 \mu s \times R \equiv \tau' R$

So, require $0.91 cm^{-1} = \frac{\lambda^2}{8\pi t_{sp}} g(\nu) R \tau'$

$$R = \frac{91 m^{-1} \; 8\pi \cdot 1ms}{(471 nm)^2 \; (250 \mu s)(6.4 \times 10^{-12} s)} = 6.4 \times 10^{27} \frac{1}{m^3 \cdot s}$$

But pump rate R is given by

$$R = W_{0\to 3} N_a \qquad (W_{0\to 3} = 644\ s^{-1})$$

$$= \frac{\lambda^2}{8\pi t_{sp}} g(\nu) \frac{I}{h\nu} \frac{g_3}{g_0} N_a$$

for $\lambda = 500\ nm/n = 294\ nm$

$t_{sp} = 100\ ns$

$g(\nu) = \frac{2}{\pi \Delta\nu_{03}} = 1.3 \times 10^{-14}\ s$

$h\nu = \frac{hc}{\lambda} = 3.98 \times 10^{-19}\ J$

$N_a = 10^{25}\ m^{-3}$

So

$$I_p = \frac{6.4 \times 10^{27} m^{-3} s^{-1}}{10^{25}\ m^{-3}} \cdot \frac{8\pi (100 ns)}{(294\ nm)^2} \frac{1}{1.3 \times 10^{-14} s}$$

$$\times \frac{2}{8} \cdot 3.98 \times 10^{-19}\ J$$

$$\boxed{I_p = 1.4 \times 10^5\ W/m^2}$$

4.

a) know absorption coefficient : $\alpha = \sigma N_a$

So $\sigma_{12} = \frac{g_1}{g_2} \frac{\alpha}{N_a} = \frac{0.1\ cm^{-1}}{10^{17} cm^{-3}} = 10^{-18}\ cm^2.$

and $\sigma_{12} = \frac{\lambda^2}{8\pi t_{sp}} g(\nu)$

For collision broadening, $g(\nu)$ is Lorentzian with

$$\Delta\nu = \frac{1}{2\pi}\left(\frac{1}{\tau_2} + 2 f_{col}\right)$$

Expect: $f_{col} \gg \frac{1}{\tau_2}$, so $\Delta\nu = \frac{f_{col}}{\pi}$

and $g(\nu_0) = \frac{2}{\pi\Delta\nu} = \frac{2}{f_{col}}$

so $t_{sp} = \frac{\lambda^2}{8\pi\sigma_{12}} \frac{2}{f_{col}} = \frac{(510\ nm/1.3)^2}{8\pi \times 10^{-18} cm^2} \frac{2}{10^{12} s^{-1}}$

$$\boxed{t_{sp} = 150\ \mu s}$$

Indeed, $f_{col} = 10^{12} s^{-1} \gg \frac{1}{t_s} = 6.5 \times 10^3\ s^{-1}$

b) For three-level system, rate equations give

$$\Delta N_0 = \frac{W_p \tau_2 - 1}{W_p \tau_2 + 1} N_a$$

$$\gamma = \sigma \Delta N$$

and. $2\gamma_t l = L + T = 0.1$ (7)

$$\gamma_t = \frac{0.1}{20\,\text{cm}} = 5\times10^{-3}\ \text{cm}^{-1}$$

need $$\Delta N_t = \frac{\gamma_t}{\sigma} = \frac{5\times10^{-3}\ \text{cm}^{-1}}{10^{-18}\ \text{cm}^2} = 5\times10^{15}\ \text{cm}^{-3}$$

so $$\Delta N_t / N_a = 0.05$$

solve, $$W_p \tau_2 = \frac{1+\Delta N/N_a}{1-\Delta N/N_a} = 1.1$$

$$W_p = \frac{1.1}{\tau_2} = \frac{1.1}{150\,\mu s} = 7.4\times10^3\ s^{-1}$$

Relate to absorption:

$$W_p = \sigma_{13}\, I_p / h\nu_{13}$$

$$\sigma_{13} = \frac{\alpha_{13}}{N_a} = \frac{10\ \text{cm}^{-1}}{10^{17}\ \text{cm}^{-3}} = 10^{-16}\ \text{cm}^2$$

so $$I_p = W_p \frac{h\nu_{13}}{\sigma_{13}} = 7.4\times10^3\ s^{-1}\ \frac{(6.63\times10^{-34}\,\text{J.s})\left(\frac{3\times10^8\,\text{m/s}}{570\,\text{nm}}\right)}{10^{-16}\ \text{cm}^2}$$

$$\boxed{I_p = 28.3\ \text{W/cm}^2}$$

c) $$I_{sat} = \frac{h\nu}{\sigma_{12}\, t_s}$$

For three level system have. ⑧

$$\tau_s = \frac{2\tau_{sp}}{1 + \tau_{sp} W_p} \qquad (S \& T \;\; 13.2-25.)$$

$\tau_{sp} W_p = 1.1$ at threshold

$$\tau_s = \frac{300\,\mu s}{1 + 1.1} = 142\,\mu s$$

$$I_s = \frac{(6.63 \times 10^{-34}\,J\cdot s)\left(\frac{3 \times 10^8\,m/s}{570\,nm}\right)}{(10^{-18}\,cm^2)\,(142\,\mu s)} = 2457\ W/cm^2$$

d) If $W_p = 2W_t$, $W_p \tau_2 = 2.2$

then
$$\Delta N_0 = \frac{W_p \tau_2 - 1}{W_p \tau_2 + 1} N_a$$

$$= \frac{2.2 - 1}{2.2 + 1} N_a = 0.375\, N_a$$

$$= 3.75 \times 10^{16}\, cm^{-3}$$

Then $\gamma = \sigma_{12}\,\Delta N_0 = 10^{-18}\,cm^2 \cdot 3.75 \times 10^{16}\,cm^{-3} = 0.0375\ cm^{-1}$

$$g_0 = 2\gamma_0 l = 0.75$$

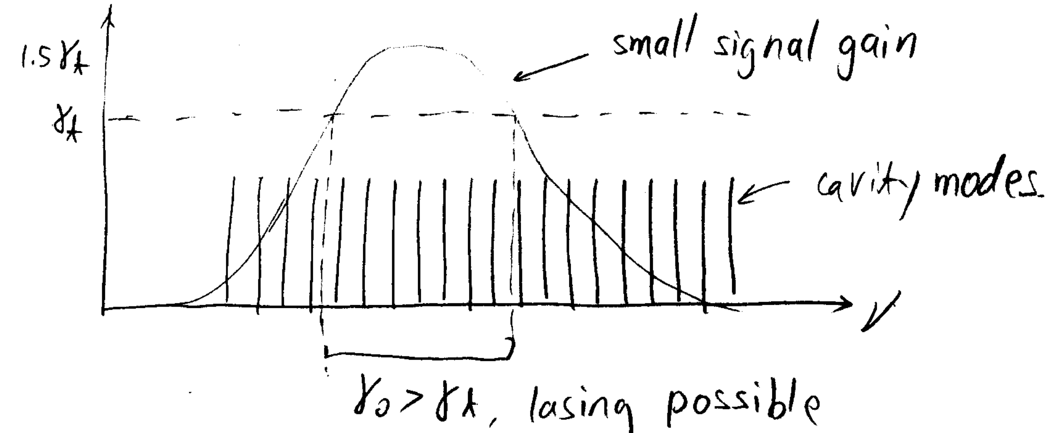
$$I_s = \frac{h\nu}{\sigma_{12}} \cdot \frac{1 + \tau_{sp} W_p}{2\tau_{sp}} = 2457 \frac{W}{cm^2} \doteq 3744\ W/cm^2$$

Then $P_{out} = \pi W^2 I_s \left(\frac{g_0}{T+L} - 1\right) T$

$$= \pi\,(100\,\mu m)^2 \times \left(3744 \frac{W}{cm^2}\right)\left(\frac{0.75}{0.1} - 1\right)(0.05)$$

$$\boxed{P_{out} = 0.38\ W}$$

5. Picture:



Doppler profile:

$$\bar{g}(\nu) = \frac{1}{\sqrt{2\pi}\,\sigma_D} e^{-\frac{(\nu-\nu_0)^2}{2\sigma_D^2}}$$

$$\sigma_D = \frac{1}{\lambda}\left(\frac{kT}{M}\right)^{1/2}$$

$$= \frac{1}{514\text{ nm}}\left(\frac{1.38\times10^{-23}\frac{J}{K}\cdot 2000\,K}{40\times1.67\times10^{-27}\text{ kg}}\right)^{1/2}$$

$$= 1.25\text{ GHz}$$

Find ν such that $\bar{g}(\nu) = \frac{2}{3}g(\nu_0)$ (that is $\gamma > \gamma_t$)

$$e^{-\frac{(\nu-\nu_0)^2}{2\sigma_0^2}} = \frac{2}{3}$$

$$(\nu-\nu_0)^2 = 2\sigma_0^2 \ln 1.5$$

$$\nu-\nu_0 = \pm\sigma_0\sqrt{2\ln1.5}$$

$$\nu = \nu_0 \pm \sigma_0\sqrt{2\ln1.5}$$

Freq. range $\Delta\nu = 2\sigma_0 \sqrt{2\ln 1.5}$

$= 1.8\,\sigma_0$

$= 2.25$ GHz

On the othe hand, free spectral range of cavity is $\nu_F = \frac{c}{2d} = 3 \times 10^8$ Hz $= 0.3$ GHz.

So expect about $\frac{2.25 \text{ GHz}}{0.3 \text{ GHz}} = 7.8$.

So $\boxed{7 \text{ or } 8}$ modes should lase.