TWO BASIC RESULTS

Time dilation

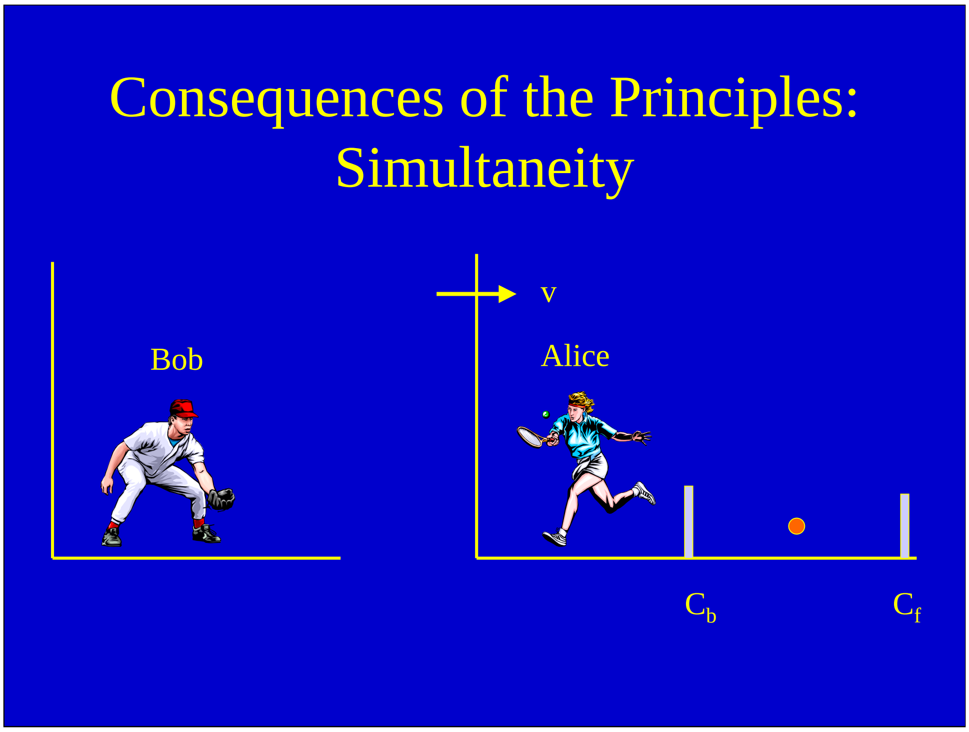
$$\Delta t(v) = [1/(1 - v^2/c^2)^{1/2}]\Delta t(0)$$

Length Contraction

$$d(v) = (1 - v^2/c^2)^{1/2} d(0)$$

These two basic results are at the core of special relativity: Moving clocks run slow, and moving meter sticks are shorter along the direction of motion. As we have seen, length contraction is a direct result of time dilation, and time dilation depends directly on the fact that all observers see light moving at the same speed c.

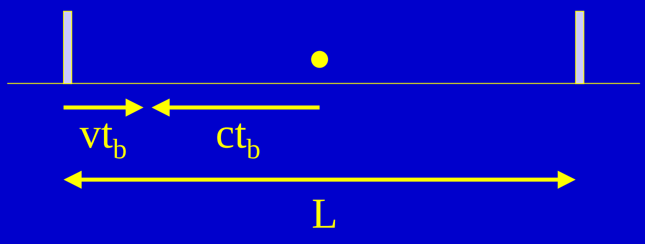
Earlier we noted that events that are simultaneous for one observer may not be for another who is moving relative to the first. We noted qualitatively that this must be the case. Now we can be quantitative about it. When two clocks are synchronized in one frame, by how much do they differ in another that is moving? This is a very important and useful result.



Here is the flash bulb halfway between markers A and B in Alice’s frame. To be quantitative about the time difference Bob will see, we place clocks C_b (the back clock according to Bob) and C_f (the front clock) at the two markers with photocells that will start the clocks when the light pulses arrive. For Alice, the two flashes arrive simultaneously, so the two clocks are synchronized.

We know that Bob observes the light pulse to arrive at C_b first. The question is, by how much? That is, by how much are the two clocks out of synchronization according to Bob?

HOW MUCH DIFFERENCE?



$$vt_b + ct_b = 1/2L(1 - v^2/c^2)^{1/2}$$

$$t_b = [1/(c + v)]1/2L(1 - v^2/c^2)^{1/2}$$

$$t_f = [1/(c - v)]1/2L(1 - v^2/c^2)^{1/2}$$

Here is a sketch of the light pulse as it is on its way toward the back clock, C_b. As the pulse travels back towards the clock, it moves forward towards the pulse. The time required for the pulse to get to the back clock is t_b.Since Bob is making these observations, we must use the contracted length $L(1-v^2/c^2)$ for the distance between the clocks. Similarly the time for the pulse going forward to get to the front marker that is moving forward, is t_f. So now we can evaluate the difference between these two times, which is the amount by which the two clocks are out of synchronization according to Bob.

t_f - t_b

$$t_f - t_b = [1/(c-v) - 1/(c+v)]1/2L(1-v^2/c^2)$$
$$= v/c^2[1/(1-v^2/c^2)]L(1-v^2/c^2)^{1/2}$$

$$t_f - t_b = (vL/c^2)[1/(1 - v^2/c^2)^{1/2}]$$

This is the time difference between the back and front clocks as seen by Bob. He knows however that Alice's clocks run slow by the factor $(1 - v^2/c^2)^{1/2}$ so he concludes that her two clocks will differ by vL/c^2.

Bob sees the back clock start before the front clock by this time interval. Knowing his special relativity he is aware that Alice's clocks run slow. So he concludes that her two clocks will differ by vL/c^2. That is, the back clock will read this amount when the front clock starts.

Note that if $L = 0$, that is if the two clocks are together, Bob and Alice will both say they are synchronized. There must be a distance between the clocks as well as relative motion to produce a disagreement about synchronization.

EXAMPLE

If $v/c = 3/5$, and $L = 10^9$ m, then

$vL/c^2 = (v/c)(L/c) = 3/5(10^9/3*10^8) = (3/5)(10/3)$
$= 2$ seconds.

For $L = 3$m, $vL/c^2 = (3/5)(3/3*10^8) = (3/5)10$ ns
$= 6$ ns.

Here are two numerical examples of the synchronization results we have just obtained. The speed v/c = 3/5 is convenient because it makes the arithmetic easy, involving integers. Using a large distance of a billion meters (a million kilometers, or 625,000 miles) we find a change in synchronization of 2 seconds.

Using a more modest distance of 3 m and the same speed, we find a change in synchronization of 6 ns. How important these changes are depends on the circumstances. We will see an example right away where the change is important for explanatory purposes, regardless of whether it is large or small.

RELATIVITY TOOLKIT

Time Dilation:

$$\Delta t(v) = [1/(1 - v^2/c^2)^{1/2}]\Delta t(0)$$

Length Contraction:

$$d(v) = (1 - v^2/c^2)^{1/2}d(0)$$

Change in Synchronization:

$$\Delta t = Lv/c^2$$

Light time of flight:

$$t = d/c$$

We have now studied all the major features of special relativity needed to solve most problems. Here are four major ideas that are all you need to handle various problems, quandries, paradoxes.

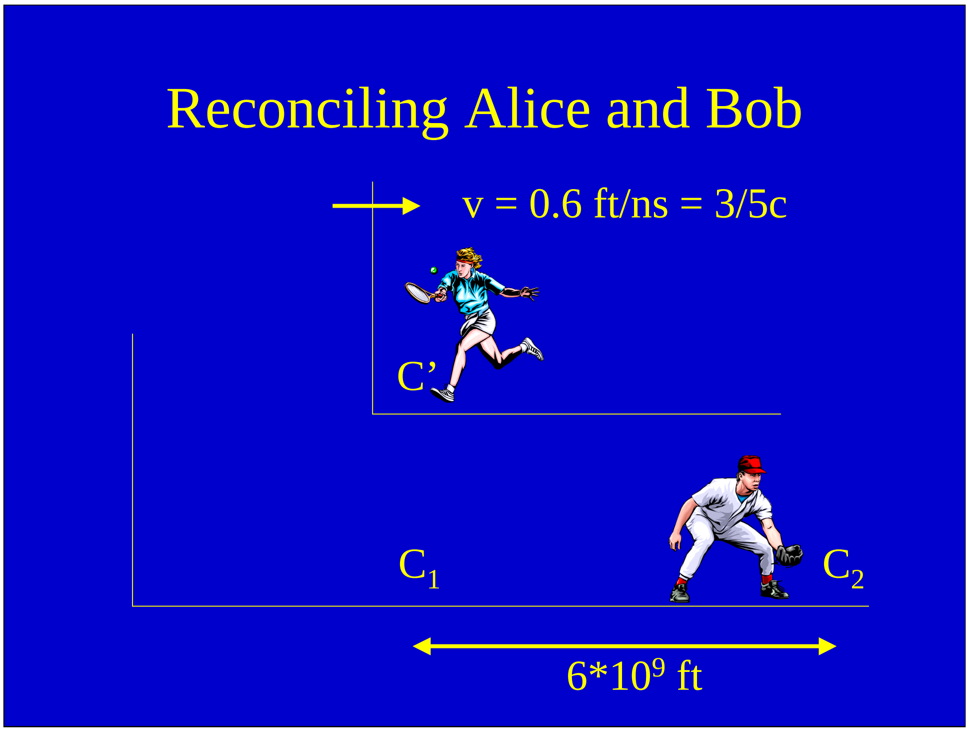
When objects move with speeds near the speed of light, new things happen that do not correspond to our common sense, or intuition. Einstein once said that common sense is that layer of prejudice that we absorb as children. You can use it to check the results of most Newton's laws problems. But it doesn't work here because we developed our intuition about movement in the world entirely at low speeds. So you must think carefully as you apply these four ideas.

Reconciling Alice and Bob

How can Alice and Bob each see the other's clocks to be running slow and unsynchronized, and agree on anything?

Suppose they both look at the same clock at the same time from the same place. Will they agree on what time it shows? Let us see how.

At this point we have Alice and Bob disagreeing on nearly everything they both observe. But if they both know about the relativity toolkit just shown, they should be able to sort things out so they understand each other. Let's look again at the two clock problem, only this time let's let Bob have the two clocks, since of course the two frames are equally valid, and we don't want to imply otherwise.



Bob has two clocks, C_1 and C_2 in his reference frame. They are located $6*10^9$ ft which is 6 light seconds apart. Alice is moving with speed 0.6 ft/ns = 3/5c and has clock C’ in her reference frame. C’ and C_1 both start at the moment C’ passes C_1.

Bob’s two clocks are synchronized and he knows it will take Alice 10 s to get to C_2 so that is what his clock reads as Alice goes past. What does Alice’s clock read as she passes C_2? $(1 – v^2/c^2)^{1/2} = (1 – (3/5)^2)^{1/2} = 4/5 = 0.8$. So her clock will read 8 s as she passes C_2. Since Alice and Bob are at the same location as their two clocks, they will agree that his clock reads 10s and hers reads 8s. How then can Alice claim that Bob’s clocks run slow?

To Alice, Bob’s clocks are not synchronized. C_1 is behind C_2 by $Lv/c^2 = (6s)(0.6) = 3.6s$. So she concludes that since C_2 reads 10s, at that instant C_1 must read 6.4s. Her clock reads 8s, so Bob’s clocks are running slow by the factor 6.4/8 = 0.8 just as they should. The change in synchronization allows them both to see the other’s clocks as running slow.

Observations

As Alice and her clock pass C_2, it reads 10s. Her clock reads $10s*(1 – v^2/c^2)^{1/2} = 8s$. Since they are at the same place, Alice and Bob agree on these two times.

Then how can Alice argue that Bob's clocks run slow? To Alice, Bob's clocks are not synchronized. C_1 is behind C_2 by $Lv/c^2 = (L/c)(v/c) = 6*0.6 = 3.6s$.

Alice concludes that as she passes C_2, C_1 must read 6.4s, and that Bob's clocks are running slow by the factor $6.4/8 = 0.8$

Bob's two clocks are synchronized according to him and he knows it will take Alice 10 s to get to C_2 so that is what his clock reads as Alice goes past. What does Alice's clock read as she passes C_2? $(1 – v^2/c^2)^{1/2} = (1 – (3/5)^2)^{1/2} = 4/5 = 0.8$. So her clock will read 8 s as she passes C_2. Since Alice and Bob are at the same location as their two clocks, they will agree that his clock reads 10s and hers reads 8s. How then can Alice claim that Bob's clocks run slow?

To Alice, Bob's clocks are not synchronized. C_1 is the back clock for her, and so C_2 is ahead of it. C_1 is behind C_2 by $Lv/c^2 = (6s)(0.6) = 3.6s$. So she concludes that since C_2 reads 10s, at that instant C_1 must read 6.4s. Her clock reads 8s, so Bob's clocks are running slow by the factor $6.4/8 = 0.8$ just as they should. The change in synchronization allows them both to see the other's clocks as running slow.

They Photograph C_1

To check all this, they both digitally photograph C_1 through telescopes. Bob's picture shows C_1 reading 4s as it must. His clocks are synchronized and they are 6 light seconds apart.

Alice's picture must show the same. The two pictures were taken at the same time from the same place. How can she reconcile her picture showing C_1 reading 4s with her assertion that at the instant she took the picture it read 6.4s?

So far Alice and Bob are uncharacteristically in agreement. They decide to push their luck and take digital pictures of C_1 just at the instant C' passes C_2 through powerful telescopes (6 billion feet is 1.1 million miles). Bob knows his picture will show C_1 reading 4s since his clocks are synchronized, C_2 reads 10s, and they are 6 light seconds apart.

What does Alice's picture show? Since the two pictures were taken at the same time from the same place, and light travels just as fast for her as for him, they must show the same thing, so her picture also shows C_1 reading 4s. How can she reconcile this with her assertion that C_1 read 6.4s at the instant she took the picture?

Fortunately she knows about special relativity.

Alice’s Explanation

1. Distance to C_1 = $4/5*6*10^9$ ft. = $24/5*10^9$ ft

2. When did the light entering her camera leave C_1?

vt ct

$$vt + ct = 1.6*10^9 t = 24/5*10^9$$

$$t = 3\text{s ago her time}$$

3. Time dilation: 3s*4/5 = 2.4 s ago clock time. So the clock read 4 + 2.4 = 6.4s when she took the picture.

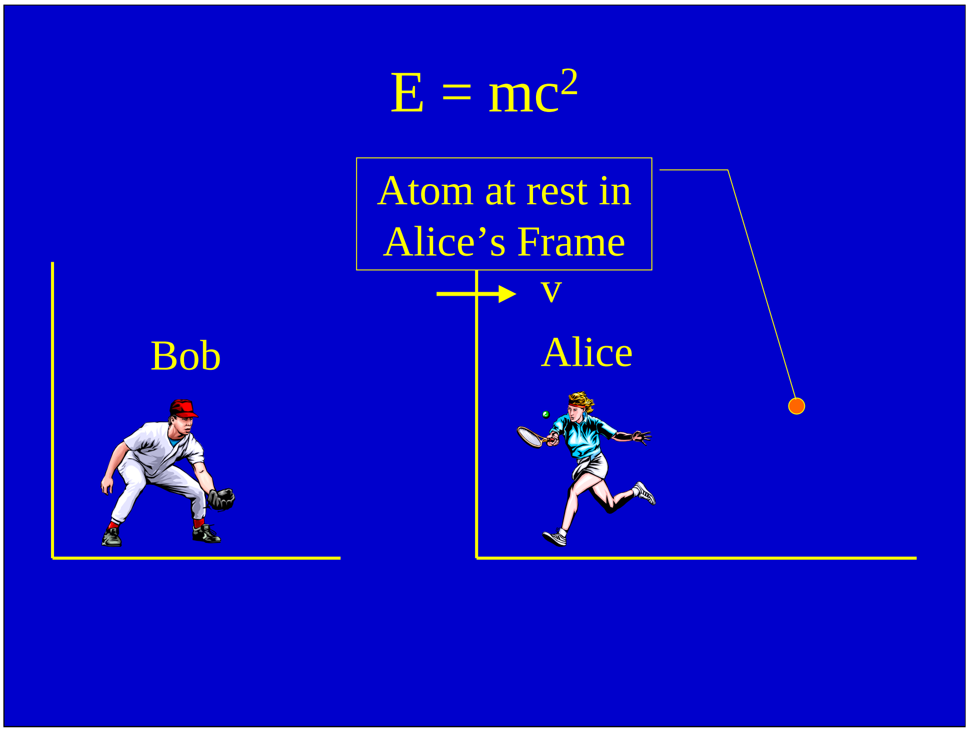
First Alice calculates her distance to C_1. This is shorter by the length contraction factor of 4/5 than Bob measures it.

Then she finds how long ago the light that entered her camera left C_1. It did not have to go the full contracted distance calculated above since it is moving away from her (and so is the back clock for her). (Here we are using c = 1 ft/ns = 10^9 ft/s.) She finds it left the clock 3 seconds ago her time.

Now she must evaluate how many seconds ticked off the moving clock during this 3 second interval in her frame. By time dilation this is 4/5 of 3 seconds, or 2.4 seconds.

So she has found that clock 1 must have read 4 + 2.4 = 6.4 seconds at the instant she took the picture while passing C_2. This is just what she concluded using the change in synchronization between the two frames. We have a completely consistent picture.

At first it seems impossible for Alice and Bob to each maintain that the other’s clocks run slow compared with theirs. But by bringing in the other necessary consequences of relativity shown in the relativity toolkit, the whole picture becomes completely consistent. If Alice and Bob think carefully, they realize they actually agree with each other!

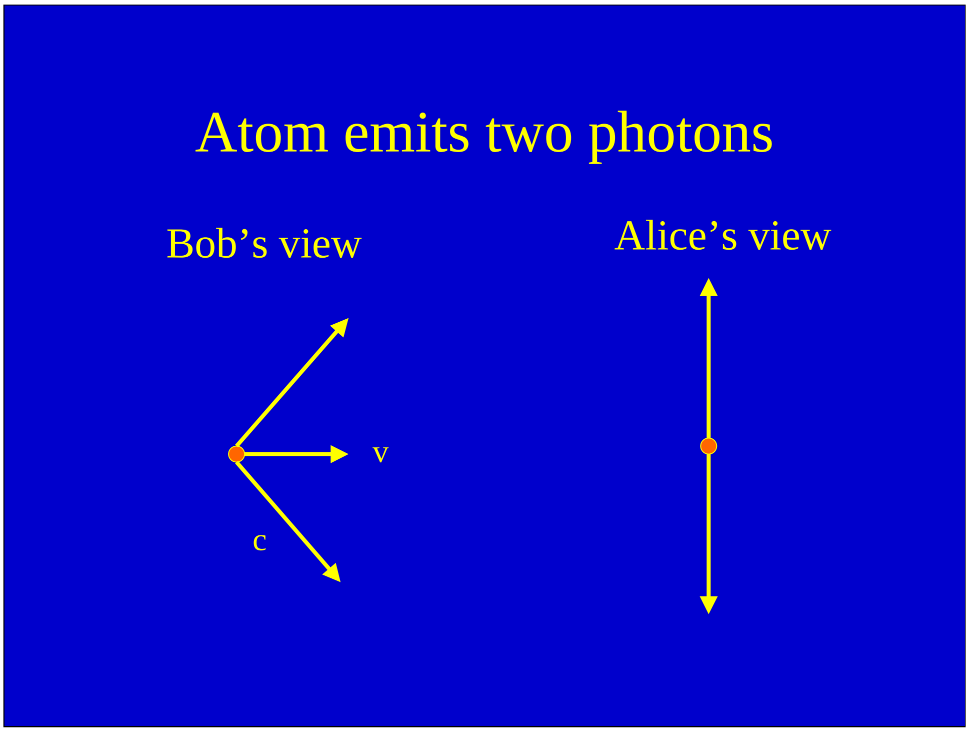


Here are Alice and Bob in their reference frames as usual. In addition we have placed an atom at rest in Alice's frame. The atom emits two photons (this means the atom was in an excited state – its electrons were not in equilibrium when we put it there. Atoms in equilibrium do not emit light). The two photons travel away from the atom in opposite directions perpendicular to the relative velocity v.

This means the energy of the atom has decreased by an amount 2hf, since as we have seen the energy of a photon is equal to hf. So energy will be conserved overall if the energy of the atom decreases by the amount of the energy of the two photons.

Since there are no external forces acting on the system, we also know that momentum will be conserved.

We have seen that we can sometimes learn important and surprising things just by carefully describing what two observers in relative motion see as an event occurs. How do Bob and Alice see the emission of the photons by the atom?



Alice sees the two photons move away from the atom in opposite directions. The atom was stationary initially, and after emitting the photons it remains stationary. Each photon carries momentum hf/c, as we have seen from the Compton scattering experiment, and since they move away in opposite directions, the atom picks up no net momentum from the photons. So energy and momentum are both conserved in an obvious way for Alice.

Bob sees the atom moving with speed v initially. After the photons are emitted, since the atom remains stationary in Alice's frame, it continues with the same speed v. But Bob sees the two photons moving away from the atom with a forward component to their speed as well as equal and opposite vertical components.

Each photon has a momentum with a forward component equal to (hf/c)(v/c). Since the atom moves with the same speed v as before, how can we arrange to have momentum conserved from Bob's point of view?

Einstein was so convinced that momentum must be conserved that he was willing to say that the mass of the atom must have changed.

Energy and Momentum of Atom

$$\Delta E_{Atom} = -2hf$$

$$\Delta(mv)_{Atom} = -2(hf/c)(v/c) = \Delta E_{Atom}(v/c^2)$$

Since v of atom remains constant, $\Delta(mv)_{Atom} = v\Delta m_{Atom}$

So $\Delta E_{Atom} = \Delta m_{Atom} c^2$

This is true for any agent that changes the energy of an atom.

In general, $E = mc^2$

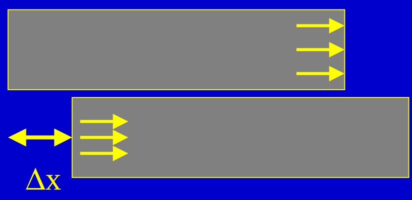
When the atom emitted the two photons, its energy decreased by 2hf. For momentum to be conserved in Bob’s frame, the atom’s momentum must have decreased since now the photons have forward momentum. But since the velocity of the atom remained the same, the only way for its momentum to decrease is for the mass to do so.

In this example it is photons that changed the energy of the atom. The result is true in general however. Any external agent that changes the energy of an atom, also changes its mass, or inertia by the amount shown above. Since a chair is made of atoms, the same result applies to chairs, rocks, etc.

For example if you pick up a 1 kg mass from the floor and put it on a table, you have increased its mass by: $\Delta m = mgh/c^2 = 10^{-16}$ kg. For nuclear reactions like the fusion reactions that power the sun, the change in mass is about one part per thousand of the mass of the protons involved. For typical chemical reactions, it is around one part per billion.

This gives us an entirely new way of viewing mass. Mass is energy. Newton thought of mass as either inertia or gravitational mass. Einstein thought of it as energy. He regarded this as his most important result.

EINSTEIN'S BOX



Isolated box has mass M, length L

The left end emits a burst of photons with energy E, momentum E/c. By conservation of momentum, this gives the box a velocity. $Mv = -E/c$. While the photons travel to the other end, the box drifts a distance $\Delta x = v\Delta t = (-E/Mc)(L/c) = -EL/Mc^2$

Einstein argued that the center of mass must remain stationary since no external forces have acted. This can only occur if the photons have an equivalent mass m given by: $mL + M\Delta x = 0$

Solving for m: $m = E/c^2$ or $E = mc^2$

This is a simple argument that Einstein used in 1906 to show that energy and mass are the same thing. A box is floating in space far from other objects. One end emits a burst of radiant energy (photons) having energy E. We know that this burst also has momentum E/c. Then while the photons travel towards the other end of the box, since momentum is conserved, the box must be drifting with velocity v given above.

When the burst gets to the other end of the box it stops drifting. Then the box has moved over an amount Δx. Einstein then argued that the center of mass could not have moved since no external forces have acted. Since the box has moved over the only way its CM could remain stationary is if the photons, moving in the opposite direction, have inertia too. The required amount is E/c^2.

Before Einstein two important conservation laws in physics were: Conservation of energy, as we have seen, and conservation of mass. What Einstein did with this relation is unite them. Now there is simply the conservation of mass/energy.

Once the radiant energy reaches the other end of the box it is absorbed and becomes heat energy. Heat energy can be used to run an engine to produce nearly any other kind of energy. So this inertia of energy is completely general. A moving golf ball has more mass than a stationary one.

Fusion Reaction in the Sun

The sun is fueled by nuclear fusion reactions. One of them is:

$$p + D \rightarrow He^3 + \gamma$$

Here p = proton, D = deuterium nucleus (1 proton and 1 neutron) He^3 is a Helium 3 nucleus (2 protons, 1 neutron), and γ is a gamma ray (high energy photon).

This is an exothermic reaction (gives off energy in the form of the gamma ray).

As an example of Einstein's famous equation let's look at a nuclear reaction that is essential for life on earth. The sun is powered by nuclear fusion reactions in which light nuclei come together to form heavier ones. A series of reactions are involved, one of which is shown above.

This reaction gives off energy in the form of the gamma ray shown on the right side of the equation. The reaction takes place deep in the sun where the temperature is around 10 million degrees Celsius. The gamma rays are absorbed. Their energy finally escapes the sun in the form of radiant energy emitted by the surface of the sun. This is just the heat and light we need for life.

Let's now look at the energy budget for this reaction.

Energy Budget for Reaction

The masses of the constituent particles are:

proton:	$1.6724*10^{-27}$ kg
deuteron:	3.3432
p + D:	5.0156
He^3:	5.0058
Difference:	0.0098

The proton and deuteron have more mass than the He^3 nucleus.

The difference is an energy of Δmc^2 = $(9.8*10^{-30}\text{kg})(9*10^{16}\ \text{m}^2/\text{s}^2) = 8.8*10^{-13}\text{J}$

To find out how much energy this reaction releases all we need to know is the masses of the particles on both sides of the equation. The proton plus deuteron, the reacting particles, have more mass than the He^3 nucleus that is formed. The difference, when multiplied by c^2 is the energy released by the reaction.

This is the energy of the gamma ray produced by the reaction.

The sun loses about 4.6 million tons per second due to reactions such as this one. This is only about one part in 10^{13} of its mass per year. It is expected to live another 10 billion years.

The domestic energy used per day in Charlottesville, when divided by $c^{2,}$ is about one tenth of a gram.

In the above fusion reaction, about one part in 500 of the initial mass is changed to available energy. For a typical chemical reaction, about one part in one billion of the initial mass is changed to available energy.