RELATIVITY TOOLKIT

Time Dilation:

$$\Delta t(v) = [1/(1 - v^2/c^2)^{1/2}]\Delta t(0)$$

Length Contraction:

$$d(v) = (1 - v^2/c^2)^{1/2} d(0)$$

Change in Synchronization:

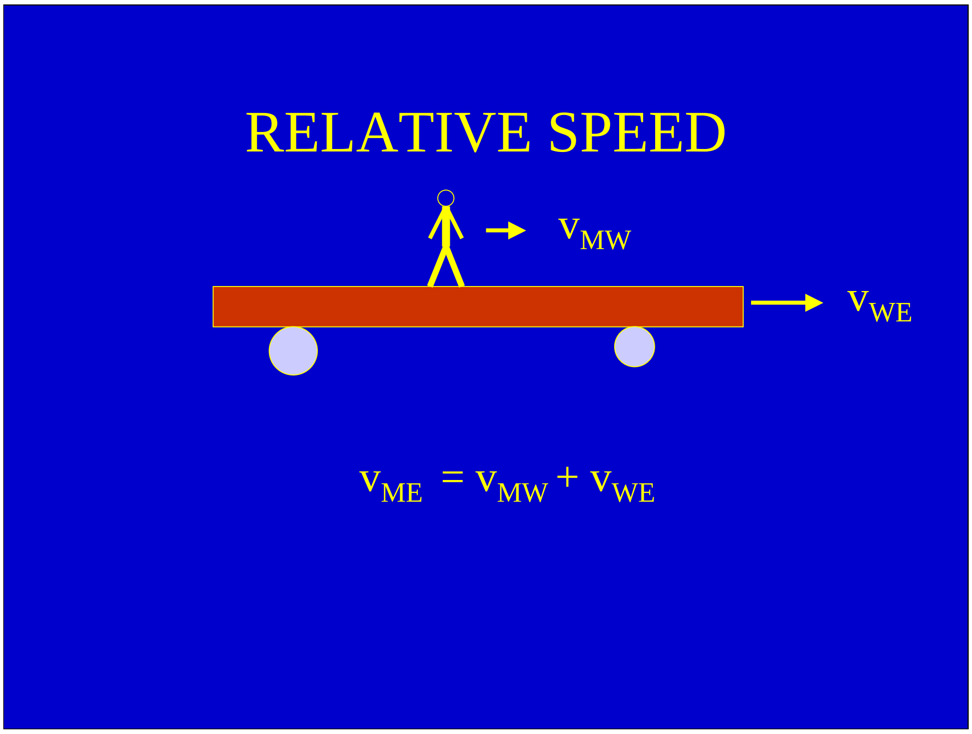
$$\Delta t = Lv/c^2$$

Light time of flight:

$$t = d/c$$

We have now studied all the major features of special relativity needed to solve most problems. Here are four major ideas that are all you need to handle various problems, quandries, paradoxes.

When objects move with speeds near the speed of light, new things happen that do not correspond to our common sense, or intuition. Einstein once said that common sense is that layer of prejudice that we absorb as children. You can use it to check the results of most Newton’s laws problems. But it doesn’t work here because we developed our intuition about movement in the world entirely at low speeds. So you must think carefully as you apply these four ideas.



Speeds are always measured with respect to an origin of coordinates, or a reference frame. Sometimes that reference frame might itself be moving with respect to something else. We saw an example of this in the Ptolemaic model of the universe – the reference frame for that model is the Earth which is itself moving with respect to the Sun.

How do we take a moving reference frame into account? The Galilean method is illustrated here. A wagon moves along the ground with a speed v_{WE}, the speed of the wagon with respect to the Earth. A man walks on the wagon with a speed v_{MW}, the speed of the man with respect to the wagon. What is the man's speed with respect to the Earth? It is simply the sum of the above two speeds.

Underlying this method of combining speeds are the assumptions that time and space are absolute. That is, the clocks on the wagon run at the same rate as those on the ground, and the length of a meter is the same for both. Clearly we need to rethink this question in the light of the consequences of special relativity. We already know that nothing is added to the speed of light when the light is emitted from a moving source. A general answer to this question must include this result.

ADDING VELOCITIES

A train, length L, moves with speed v relative to the ground, while a walker on the train walks forward with speed u relative to the train. What is the speed of the walker relative to the ground?

Clocks C_1 and C_2 are placed at the rear and front of the train respectively, and are synchronized on the train.

A ground observer has clock C'. C_1 and C' both start when they are opposite each other.

Here we have a train moving to the right with speed v relative to the ground. A walker on the train moves with speed u relative to the train. Allowing for the possibility that both speeds are relativistic, what is the speed of the walker relative to the ground?

We will need to measure both times and distances, so we put synchronized (on the train) clocks at the rear and front of the train. A ground observer's clock and C_1 start when they are opposite each other which is when the walker begins his walk.

So C_1 and C' both read 0 at the start of the walk.

ADDING VELOCITIES II

$C_1 = C' = 0$ at the start of the walk.

At the end of the walk, $C_2 = L/u$ according to walker. Ground observer agrees.
At that instant, the ground observer says $C_1 = L/u + Lv/c^2$ due to the synchronization change. This is the duration of the walk measured by train clocks. So the ground observer says the duration measured on his clock was:

$$t_w = [L/u + Lv/c^2]/(1 - v^2/c^2)^{1/2}$$

The ground observer's clock and the rear train clock both start at zero when the walk begins. When the walker arrives at the front of the train, he will see C_2 reading L/u. The ground observer agrees since two simultaneous events at the same place are simultaneous for all observers.

At that instant however, the ground observer will say that $C_1 = L/u + Lv/c^2$. The second term is just the synchronization change that occurs when both relative velocity and spatial separation along the direction of motion are present. This is the duration of the walk according to the ground observer, but measured by train clocks.

Since the ground observer knows the train clocks run slow, he corrects for that and so says the duration measured by his clock is as shown above.

ADDING VELOCITIES III

How far does the walker go as seen by the ground observer?

Distance train traveled plus length of the train.

$$d = vt_w + L(1 - v^2/c^2)^{1/2}$$
$$= [L(1 + v/u)]/(1 - v^2/c^2)^{1/2}$$

So the walker's speed relative to the ground is:

$$d/t_w = [u + v]/(1 + uv/c^2)$$

Speed of light from a moving source: "c + v"

$$= [c + v]/(1 + v/c) = c[c + v]/(c + v) = c$$

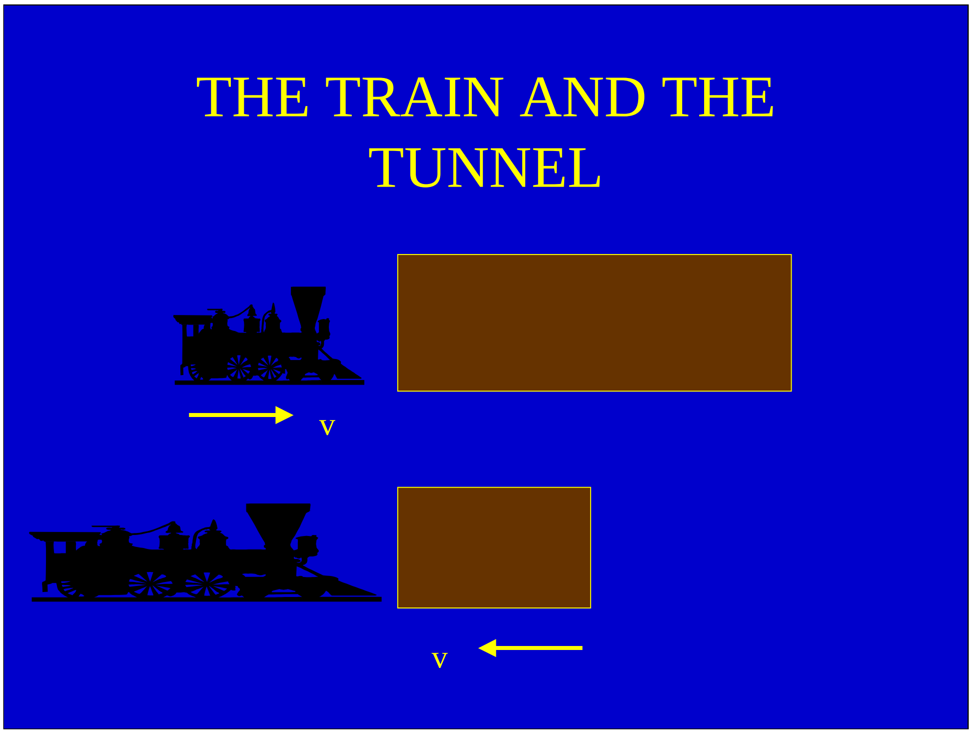
Now we know the duration of the walk, so all we need is how far the walker went according to an observer on the ground. The answer is simple: He went the distance the train traveled plus the length of the train. This is, of course, the train length as seen by a ground observer and so is contracted by the usual factor compared with the length as seen by observers on the train.

Then the walker's speed according to ground observers is just the distance he went divided by the time required for the walk.

Note that this expression reduces to the simple addition of velocities when they are small compared with c.

If we use this formula to find the speed of light emitted from a moving source, we find the result is just c as it should be, regardless of the value of v.

To obtain this result we used three of the four elements of our relativity toolkit.



This is a famous special relativity paradox that appeared early when the theory was still quite controversial. A paradox is a proposition that seems self-contradictory, and this is a good example.

A train and a tunnel through which it might pass have the same length when in the same reference frame. This train is capable of going at relativistic speeds in spite of its old fashioned appearance. Let’s say it moves down the track at a speed that produces a length contraction of a factor of two. ($v/c = 0.866$)

Here is the paradox. An observer in the tunnel, the train pirates, will say that the train is half as long as the tunnel, so he could slam the entrance and exit doors simultaneously trapping the train in the tunnel.

On the other hand, an observer on the train says the tunnel is half as long as the train. So the front end will emerge from the exit door before the rear of the train enters the front door. Therefore the pirates cannot trap the train.

How do we resolve this paradox?

TRAPPING THE TRAIN

The train will be trapped if the entrance door closes after the rear of the train passes it, and the exit door closes before the front of the train gets to it.

As with every game, there must be rules. Here is a reasonable definition of what is meant by trapping the train. This statement presumes the train pirates simultaneously close the entrance and exit doors. They could use other strategies, such as leaving the exit door closed the whole time, which would require slightly different rules.

TRAIN AND TUNNEL II

The pirates know the train schedule and arrange for a flash of light to be created at the center of the tunnel at just the right time. The doors of the tunnel will then close simultaneously trapping the train completely in the tunnel.

The engineer on the train sees the exit door close a time Lv/c^2 before the front door. So the train has time to move fully into the tunnel before the front door closes. Both observers agree that the exit door closes before the train hits it, and that the front door closes after the train enters.

No paradox, but a completely destroyed train and tunnel.

The pirates will have to very carefully arrange the timing of the closing of the doors. Fortunately for them, the trains in their country run on time with nanosecond precision. So with a flash of light from the center of the tunnel they can cause the doors to close simultaneously at the right time so the train is enclosed. This means that the entrance door closes only after the rear of the train passes it, and the exit door closes before the front of the train hits it.

The engineer on the train does not see the closing of the doors as being simultaneous. As we have discussed in the last lecture, he will see the exit door close first. The time difference is Lv/c^2. So if the pirates time their door closings just right, the exit door will close before the train hits it, and the entrance door will close only after the rear of the train passes it, entering the tunnel.

If the engineer knows special relativity, he will spend the last few nanoseconds of his life knowing that he and the pirates agree on the sequence of events that led to their demise.

This understanding is a direct result of two of our relativity toolkit elements: length contraction, and change in synchronization between different reference frames.

When the train hits the exit door it will cut a clean hole in it having the shape of the cross section of the train. This is because the stresses in the door material are transmitted at the speed of sound, and the train is traveling at nearly the speed of light. The KE of the shrapnel, assuming only 10 kg is thrown around, will be of order of 10^{18} J, roughly equivalent to 200 megatons of TNT.

THE TWINS

Twins Alice and Bob say a fond farewell. She is going on a rocket to alpha centauri, 4 light years away. She then immediately turns around and returns. Her rocket has a speed of 0.6c, so the trip will take 96/0.6 = 160 months earth time. Bob says that because her clocks run slow she will only age 4/5 of 160= 128 months, and so will be 32 months younger upon return. Twins no longer!

But how does this look from her point of view? Bob moves at the same speed relative to her, so shouldn't she think he will be younger upon her return?

In this case there is a real difference in aging for the twins. The lack of symmetry comes from the fact that Alice is not in an inertial system during starting, turnaround, and stopping. Bob remains in an inertial (unaccelerated) frame the whole time.

The twin paradox is another classic of special relativity. In this case the paradox is resolved by showing that one twin does age more than the other. If both understand special relativity they will agree on this.

The lack of symmetry between the two observers is due to the fact that Alice, the traveling twin, moves between different unaccelerated reference frames during the trip. First she is on Earth. Then in a rocket speeding away. This change requires acceleration. When she turns around at alpha centauri, she changes again. And finally, to rejoin Bob she must decelerate at the end of the trip.

All this while Bob remains in a single inertial system on Earth.

Twins send monthly light flashes

How often does Alice, according to her clock, receive flashes from Bob as she moves away. How often does she receive flashes during her return trip?

How often does Bob receive flashes from Alice during her journey out and back?

By counting received flashes, each twin can monitor the other twin’s aging.

We need a way for the twins to keep in touch that will allow them to monitor each other’s aging. They agree to send each other one light flash each month, according to the sender’s clock. They can then count up the flashes sent by their twin, and monitor the other twins’ aging.

Then it will be a simple matter of bookkeeping to determine whether one ages more and if so by how much.

Observed Time between Flashes

In the last lecture we found that Alice and Bob could agree if they made enough observations and thought about it. In that situation, Alice moved past Bob at 0.6c, the same speed as Alice's rocket. As she passed Bob's first clock, both his clock and hers read zero. When she got to his second clock, hers read 8s, and a picture she took of his first clock indicated 4s.

We conclude that Alice's observation of any Earth clock from her receding rocket will reveal it to be running at half speed. Bob's flashes will arrive every two months according to her clock.

Similarly, Bob will receive flashes from Alice, as she recedes, every two months according to earth clocks.

When we reconciled Alice and Bob in the last lecture, she was moving past him at 0.6c, the same speed we are using for Alice's rocket. When she passed his first clock, both his and hers read zero. When she got to his second clock, hers read 8s, and a picture of his first clock showed it at 4s.

If she had continuously watched his first clock, she would have seen it running at half speed the whole time. This is due to a combination of time dilation and the fact that she is getting farther away all the time, so it takes light longer to reach her.

We conclude that a receding speed of 0.6c results in the other frame's clocks appearing to run at half speed.

So in the twin problem above, both Bob and Alice will receive each other's flashes once every two months, as long as Alice continues to move away from Earth.

Watching an Approaching Clock

If Alice had watched Bob’s second clock as she approached it, how would she have described its rate? If, as she passed Bob’s first clock, she had taken a picture of it, what would she have seen? The first clock then read 0, and the second was synchronized with the first, so Bob would say the second clock reads –6s since it is 6 light seconds away. Alice’s picture, taken from the same place at the same time, must agree.

So Alice would have seen Bob’s second clock go from –6s to +10s while her clock went from 0 to 8s. She would see it running at double speed.

Referring once again to our example in the last lecture, we now ask what Alice would have seen if she had followed Bob’s second clock as she approached it. So we have her repeat the picture taking exercise, only this time when she was next to the first clock, she takes a picture of the second one.

We argue as before that her picture must agree with one taken by Bob since it is taken from the same place at the same time. And we know that Bob’s picture must show the second clock reading –6s since his first clock then reads zero and they are 6 light seconds apart.

So this means that Bob’s second clock ticked off 16s while her clock ticked off 8s. The clock she is approaching is running at double speed.

Just as above, this difference is due to a combination of time dilation and the fact that she is getting closer to the clock so it takes less and less time for the light from it to get to her.

Now we can apply this result to the twin problem.

Bookkeeping for the Twins

Alice measures the distance to Alpha Centauri to be 4/5*4 = 3.2 light years for a trip length at 0.6c of 64 months. On the way out she will receive 32 flashes from Bob. On the way back she will receive 128 flashes, for a total of 160 flashes. This is just the length of her trip in earth months. Bob ages the expected amount.

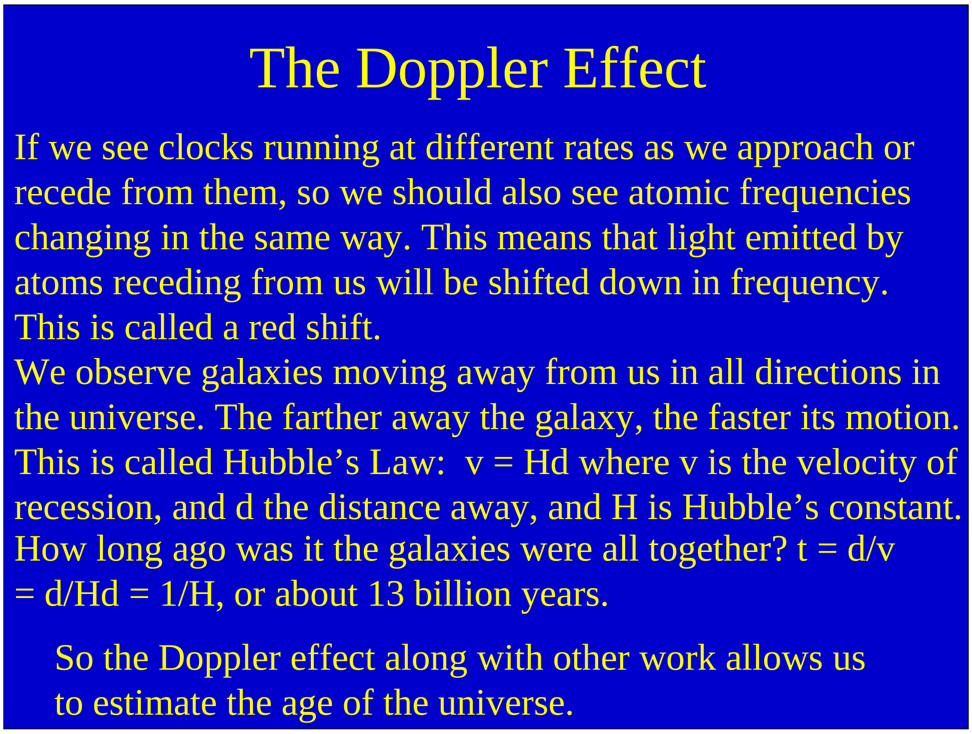
Bob sees Alice's flashes arrive every two months as she recedes. But he doesn't see her turn around until 4 years = 48 months after she does so. So during the first 80 + 48 = 128 months he receives 64 flashes. He then sees her aging at twice his rate during the final 32 months. So he counts 64 + 64 = 128 months.

She has aged 32 months less than he.

Now we are ready to add up the totals to see who aged how much. Alice calculates the contracted distance to Alpha Centauri to be 0.8*4 = 3.2 light years. At her speed the trip out will take 64 months. So she receives 32 flashes from Bob on her way out. The return trip also takes 64 months, during which she receives 128 flashes for a total of 160 flashes. This is just the length of the trip in Earth months as we have seen, so Bob ages the expected amount.

Bob sees Alice's flashes arrive every two months on her way out. But his rate from her does not double immediately as she turns around since it takes light 4 years to get back to Earth. So during the first 80 + 48 = 128 months he receives 64 flashes. Then during the final 32 months he receives another 64 flashes for a total of 128. She has indeed aged less than he by 32 months.

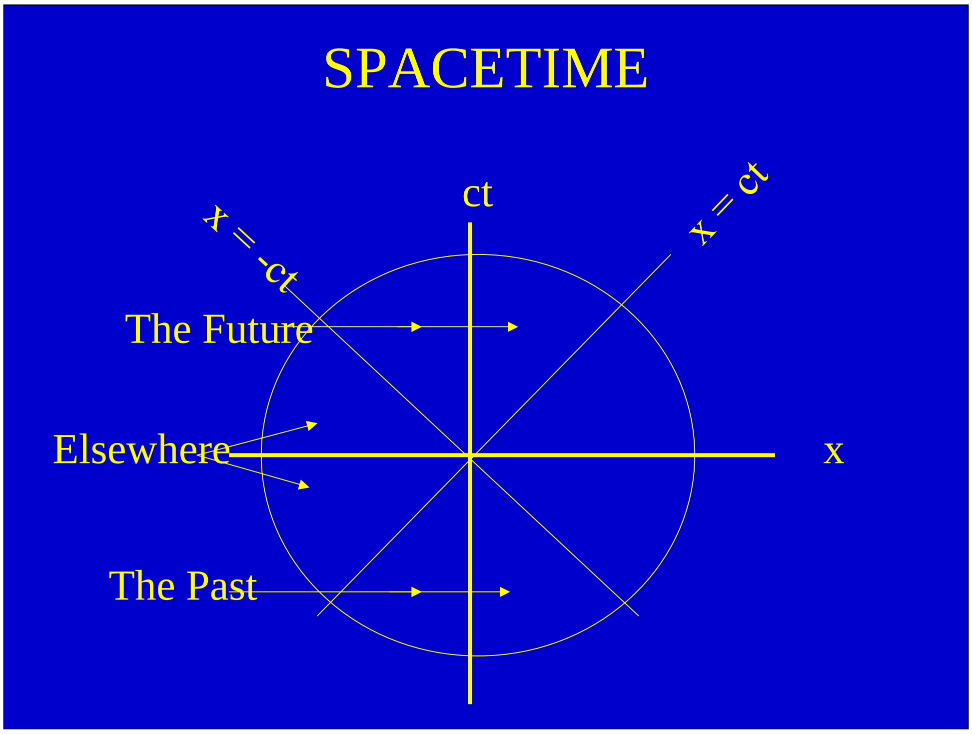
This difference is a consequence of all four of our relativity toolkit elements.



The kinematics we went through to show that clocks moving towards or away from us change their rates is general, and applies to all kinds of clocks. Included are atoms themselves. An atom receding from us emits light with lower frequencies than an atom in our reference frame. This is called a red shift since red visible light has a lower frequency than blue visible light. The effect is general, and applies outside the visible range as well as in it.

In the 1920's Edwin Hubble established that more distant galaxies are moving away from us at larger speeds. The speed measurements were done with the Doppler effect. These results eventually gave rise to the Big Bang model of the universe in which at some time in the past, all galaxies were together.

The time back to that beginning can be estimated from Hubble's Law. It's accurate value is still debated as well as whether other effects may need to be included for a careful understanding of the expansion of the universe.



In 1907 Hermann Minkowski presented a four-dimensional version of the Special Theory of Relativity. Here is a two-dimensional picture illustrating some of its features. Above we have one space dimension, x, plotted horizontally, and time, plotted vertically. Time has been multiplied by c to give it the same dimension as x. The center of the diagram is the present.

Immediately above the present is the future, and immediately below is the past. The diagonal lines correspond to light rays traveling along +x and –x directions.

Nothing can move faster than light, so the quadrants to the right and the left are inaccessible from the present. These are referred to as “elsewhere”.

Using the above approach allows many results to be presented geometrically or pictorially. This is useful in two dimensions as above, or in three dimensions (two space and time) in which the light ray lines become cones. But no one knows how to visualize the full four dimensional presentation. It is still useful for calculations, but not for visualization.