

Name _____ Date _____ Partners _____

Harmonic Motion and the Pendulum

OVERVIEW

A body is said to be in a position of stable equilibrium if, after displacement in any direction, a force restores it to that position. If a body is displaced from a position of stable equilibrium and then released, it will oscillate about the equilibrium position. We shall show that a restoring force that is proportional to the displacement from the equilibrium position will lead to a sinusoidal oscillation; in that case the system is called a **harmonic oscillator** and its motion is called a **harmonic motion**. The simple harmonic oscillator is of great importance in physics because many more complicated systems can be treated to a good approximation as harmonic oscillators. Examples are the pendulum, a stretched string (indeed all musical instruments), the molecules in a solid, and the atoms in a molecule.

The example that you will be studying in this session is the pendulum: it has a position of stable equilibrium and undergoes a simple harmonic motion for small displacements from the equilibrium position. We will first analyze the motion theoretically before testing the theory experimentally.

THEORY

A simple pendulum consists of a mass m suspended from a fixed point by a string of length L . If the mass is pulled aside and released, it will move along the arc of a circle, as shown in Fig 1. Let s be the distance from the equilibrium position measured along that arc. While the force of gravity, mg , points downward, only its tangential component along the arc, $F_{\text{tan}} = -mg \sin(\theta)$, acts to accelerate the mass. The minus sign indicates that F_{tan} is a **restoring force**, i.e. one that points in a direction opposite to that of the displacement s . From Newton's second law, $F_{\text{tan}} = ma$, we then get:

$$-mg \sin \theta = ma \quad (1)$$

By definition $a = \frac{d^2s}{dt^2}$ so $\frac{d^2s}{dt^2} = -g \sin \theta$.

We can easily express s in terms of θ by noting that $s = L\theta$, hence

$$\frac{d^2s}{dt^2} = L \frac{d^2\theta}{dt^2} \text{ and thus}$$

$$\frac{d^2\theta}{dt^2} = \frac{-g \sin \theta}{L} \quad (2)$$

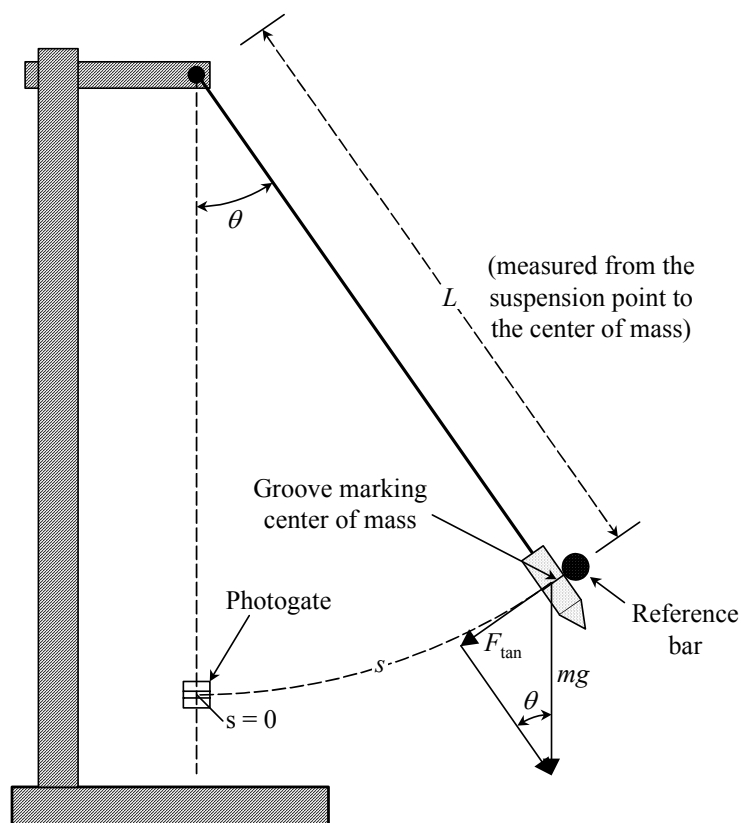


Fig. 1. Simple pendulum, schematic.

This second order differential equation is the **equation of motion** for the pendulum. It looks deceptively simple but is actually quite difficult to solve. For many applications, including ours, a simplified approximation is sufficient. We can obtain the approximate solution by using the series expansion $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$ of the sine (valid if θ is measured in *radians*). From this series expansion, we see that we can approximate $\sin \theta$ by the angle θ itself, as long as we keep θ so small that the higher order terms are much smaller than θ . By using $\sin \theta \cong \theta$ (the so-called “small angle approximation”), Eq. (2) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad (3)$$

This is the equation of motion of a simple harmonic oscillator, describing a simple harmonic motion. It is solved for arbitrary values of the **amplitude**, θ_{\max} , and the **phase**, δ , by:

$$\theta = \theta_{\max} \cos(\omega t + \delta) \quad (4)$$

If you substitute Eq. (4) into Eq. (3), you will find that the **angular frequency**, ω , is *not* arbitrary. It must have the value

$$\omega = \sqrt{\frac{g}{L}} \quad (5)$$

The cosine oscillates between -1 and 1; hence, according to Eq. (4), the angle θ will oscillate over time between $-\theta_{\max}$ and θ_{\max} . The time to execute one complete cycle is called the **period** T and is related to ω by

$$T = \frac{2\pi}{\omega} \quad (6)$$

The speed of the mass is given by $v = \frac{ds}{dt}$ or, since $s = L\theta$, by $v = L\frac{d\theta}{dt}$. If we substitute θ from Eq. (4) we obtain

$$\begin{aligned} v(t) &= -\theta_{\max} L\omega \sin(\omega t + \delta) \\ &= \theta_{\max} L\omega \cos(\omega t + \delta + \pi/2). \end{aligned} \quad (7)$$

Comparing Eq. (7) with Eq. (4), we see that the speed v oscillates with the same frequency as the angle θ but is out of phase with θ by $\pi/2$. Clearly this makes sense; the speed is zero when the angle is greatest. What is the significance of θ_{\max} and δ ? They are not determined by the previous equations but are fixed by the initial conditions. θ_{\max} is the maximum value of θ during the motion. It is determined by how far the pendulum was pulled back initially. The phase δ depends on when the clock was started that is used to measure t .

NOTE: What *is not* in an equation is often as revealing as what *is* in it. Eq. (5) does *not* contain the *mass* of the pendulum. This means that the frequency is independent of the mass. That can be true only if the mass on the left side of Eq. (1) cancels the mass on the right side. You might think that this is obvious, but it is not. The mass on the left side is a measure of the resistance that the pendulum bob offers to acceleration, which we call the **inertial mass**. The mass on the right side is a measure of the gravitational force acting on the bob, and we call it the **gravitational mass**. While one would expect the period to remain unchanged if one replaces the bob with one of the same material but a different mass, it cannot be taken for granted that the ratio of inertial to gravitational mass is the same for all materials. Newton already worried about this point and did an experiment to uncover a possible difference, but found none. Modern experiments have established that changing the material of the bob changes the ratio of two kinds of mass by no more than 1 part in 10^{11} . It was this observation that led to Einstein's *general theory of relativity*.

The materials you will need are:

- photogate timer and accessory photogate
- large and small support stands
- thread (string), meter stick, protractor
- pendulum string clamp
- brass and aluminum pendulum bobs
- digital mass scale

Helpful hints

- If you attach a bob to a new piece of string, the string will unravel somewhat, changing its length while doing so. **Wait until the bob has stopped untwisting the string** before you start a measurement.
- To make sure that you **start successive measurements with the same amplitude**, mount a steel rod (the reference bar) at an appropriate distance from the equilibrium position and start each measurement with the bob touching the rod.

INVESTIGATION 1: MASS INDEPENDENCE OF PERIOD

In this activity you will measure the period of a simple pendulum.

Activity 1-1: Period of Aluminum Bob

If we combine Eqs. (5) and (6) and note that $\omega = 2\pi/T$, we obtain

$$T = 2\pi \sqrt{\frac{L}{g}} \quad (8)$$

Note that the mass of the bob does not appear in this expression. To see if this is true, we want to *measure* the periods of an aluminum and a brass bob having different masses, but having the same length.

Question 1-1: Discuss the conditions that need to be met to test Eq. (8). Discuss, for example, the initial angle of displacement. What are its limitations? Why should the initial angle always be the same?

NOTE: To make the pendulums the same length suspend them side by side in the same clamp (*held* between the clamp and the bar) and adjust their lengths so that the center-of-mass grooves of the two bobs are at exactly the same level. **Have your instructor check your setup before proceeding!**

1. Mass both bobs using the digital mass scale.

Mass of aluminum bob: _____ g Mass of brass bob: _____ g

2. Set the pendulum lengths (measured from the pivot point to the line marking the bob's center of mass) to about 90 cm. Record the measured length:

Pendulum length: _____ cm

Prediction 1-1: What do you predict the period of the pendulum motion to be? Use the local value of the acceleration of gravity $g = 9.809 \text{ m/s}^2$. Show your calculation.

3. Move the brass bob out of the way by draping it over the stand, and measure the period of the aluminum bob. To make sure that you always start with the same amplitude, use a reference bar less than 10 cm from the equilibrium position. The timing light is best placed at the equilibrium position $s = 0$. Align the photogate such that the center of the bob breaks the "beam". The timer should be in the **pendulum mode** at 0.1 ms resolution. In pendulum mode, three triggers are necessary. The first trigger starts the timer. (In this case, the bob passing through the photogate the first time causes this first trigger.) The second trigger is ignored by the timer. (The bob swinging back through the photogate causes the second trigger.) The third trigger stops the timer (which corresponds to the bob passing through the photogate a third time – after one complete period). If the memory switch is on, additional triggers will not change the displayed time unless you push the memory switch to READ. Practice releasing the bob a few times.
4. Pull back the bob and release it. Wobbling will result in a poor measurement. Try to reduce the wobbling by holding the bob near the center of mass groove.
5. Measure the period for the aluminum bob and put your results in Table 1-1. Take as many measurements as are necessary (you do not have to fill up Table 1-1 unless you think it is necessary).

Table 1-1 Period of Pendulum

| Trial | Aluminum Time (s) | Brass Time (s) |
|-------|-------------------|----------------|
| # 1 | | |
| # 2 | | |
| # 3 | | |
| # 4 | | |
| # 5 | | |
| # 6 | | |

Activity 1-2: Period of Brass Bob

1. Hang the bobs side by side once again and see if they are still the same length. If the length has changed, you may need to measure the length again and repeat your measurements. Move the aluminum bob aside so you can measure the period of the brass bob.
2. Measure the period of the brass pendulum and enter your data into Table 1-1.

Question 1-2: What are your experimental errors in the pendulum measurement? Discuss them and make estimates here:

3. Average your measurements of the period for each mass and find the **probable error**. Excel and most standard deviation calculations return the sample standard deviation (Equation (11) in Appendix D). The *sample standard deviation* represents the statistical error of each given measurement. The *probable error* is the uncertainty in the **mean** of m measurements. See Equation (12) in Appendix D for the probable error.

Aluminum pendulum period (mean): _____ s Probable error _____ s

Brass pendulum period (mean): _____ s Probable error _____ s

Question 1-3: Is there a significant difference between the two periods? How do you know?

Question 1-4: How well do your measurements for the period agree with Prediction 1-1? Explain any differences.

INVESTIGATION 2: DEPENDENCE ON ANGLE AND LENGTH

Activity 2-1: Dependence on Angle

The simple theory says the period does not depend on the amplitude θ_{\max} , but that statement is based on the assumption that θ is small. In this experiment we want to investigate the dependence of the period on the initial angle θ . We want to see if there is a dependence on the magnitude of the initial angle and to see the effect of assuming small angles in this measurement.

1. You want to use the brass bob and measure the period T for several initial angles θ as shown in Fig 1. Make sure that you have aligned the protractor properly with the bob in its equilibrium position.

DO NOT ALLOW THE BOBS TO STRIKE THE PHOTOGATE, ESPECIALLY DURING THE LARGER OSCILLATIONS!!

2. Use $L \approx 75$ cm and $\theta = 5^\circ, 10^\circ, 20^\circ, 30^\circ,$ and 40° . For each angle, measure T three times and make sure the results are consistent. If not, remeasure until they are and average the results.

Length of Pendulum: _____ cm

| Angle | Period (s) | | | | | | |
|-------|------------|---------|---------|---------|---------|------|--------------------|
| | Degrees | Trial 1 | Trial 2 | Trial 3 | Trial 4 | Mean | STD. Error in mean |
| 5 | | | | | | | |
| 10 | | | | | | | |
| 20 | | | | | | | |
| 30 | | | | | | | |
| 40 | | | | | | | |

Question 2-1: Describe what you observe in the raw data. Is the period constant as the angle increases?

3. Use Excel to plot your average period T versus θ . Your plot should show one half of a shallow parabola opening upward. **Print** your graph.

Activity 2-2: Dependence on Pendulum Length

Question 2-2: Note that in this experiment we have had you repeat measurements. Why do you think this is so? In general how well have your measurements agreed?

1. You will now measure T for several lengths, say 30, 60, and 90 cm for one angle: choose 5° for θ .

Question 2-3: What criteria should be used when determining what initial angle to use for this experiment? Discuss these criteria.

2. Take as many trials as you think are necessary at each length and enter your data into the following table. Average your results. It is not necessary for you to have the lengths set right at 30, 60, and 90 cm. Set it close and measure the actual length. Note: You can use the “brass” data from activity 1-2 for the “90 cm” column.

| Trial | Period T (s) for each length L (cm) | | |
|-------------------------|---|-------|-------|
| Approximate length: | 30 cm | 60 cm | 90 cm |
| Actual length: | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| Mean | | | |
| Std, Error Mean | | | |
| T^2 | | | |
| Probable Error in T^2 | | | |

3. Plot the *square* of the period against L using Excel. Fit a straight line to the data. **Print** a copy for your report.

Question 2-4: From your fit to T^2 vs. L , calculate g .

$$g = \text{_____} \text{ m/s}^2$$

INVESTIGATION 3: CONSERVATION OF ENERGY

The total energy, i.e. the sum of potential and kinetic energy, should be a *constant*. On the other hand, both kinetic and potential energies change with time. To check the conversion of potential energy into kinetic energy, we want to measure the maximum speed v_{\max} of the bob as it passes through the equilibrium position (see Fig. 2). Conservation of energy requires that:

$$\frac{1}{2}mv_{\max}^2 = mgh = mg(h_2 - h_1) \quad (9)$$

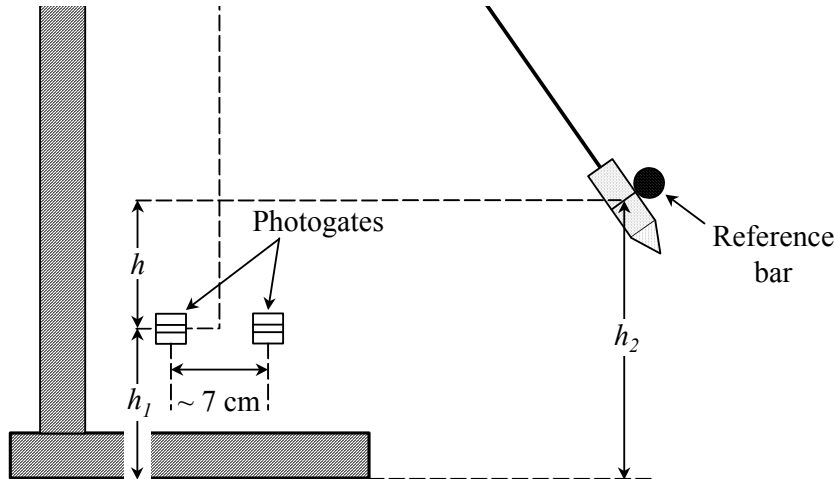


Fig. 2. The maximum speed can be obtained by measuring the time it takes the bob to travel about 7 cm at the pendulum's equilibrium position.

Activity 3-1: Measurement of Velocity

To measure the maximum velocity of the bob, we will measure how long it takes the bob to pass between two photogate timers. We can measure the time very accurately, but we must also know the distance between the timers.

Question 3-1: We claim that a distance of about 7 cm between timers is better than 3 cm. Discuss with your lab partners why this is the case. Think about possible uncertainties in the measurement. Hint: which distance is best for determining the velocity at the bottom of the arc? Hint: which distance minimizes the uncertainty in knowing the distance between the photogate timers? Which are the most important? Discuss.

1. Make the distance of the pendulum L about 80 cm or so.

Length of pendulum _____ cm

2. In order to measure the bob velocity, we must know the distance between the photogate light signals. Experience has taught us that it does not work to simply measure the distance between where the light diodes appear to be. The light paths are not predictable. Therefore, we must measure the *effective width* (Δs) of the photogate. **Your instructor will have to show you how to do this.** Wrap aluminum foil on one end of the clear plastic ruler and move the ruler along a fixed rod to determine the point at which each photogate is triggered. You can see a reflection of the red LED on the bottom side of the photogates on the table on another piece of aluminum foil that you lay flat on the table. Do this three times and average your results. Have more than one student do it.

Table 3-1 Determination of Photogate Effective Width

| Trial | Left Side | Right Side | Effective Width Δs |
|---------|-----------|------------|----------------------------|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| Average | | | |

3. If you use a sufficiently large amplitude (large L and θ), you may assume that the *average speed* between the photogates (which is what you measure) is the same as the *maximum speed* that you need in Eq. (9).
4. Set the photogate timer to pulse mode. Look at the diagram in Fig. 2. Measure the height h_1 of the bob at equilibrium with the meter stick and write the value in Table 3-2. Make sure the photogate timers are aligned the same way. **Be very careful to not let the bob hit anything!**

Height h_1 _____

5. Perform this measurement with the bob released from three widely varying heights h_2 (approximately equally spaced between limits of 25 - 55 cm). (Use of the reference bar shown in Fig. 2 may not be necessary). Use a meter stick to measure the height h_2 each time you released the bob. Write this value down in Table 3-2. Do this three times for each height h_2 and write the time difference Δt in Table 3-2.

Table 3-2 Measurement of T for Heights and Comparison of Energy

| | | | |
|------------------------------------|--|--|--|
| Height h_2 | | | |
| Height h_1 | | | |
| Height h | | | |
| Trial #1 Δt | | | |
| Trial #2 Δt | | | |
| Trial #3 Δt | | | |
| Average Δt (s) | | | |
| Velocity $v = \Delta s / \Delta t$ | | | |
| Bob Mass (kg) | | | |
| Kinetic Energy (J) | | | |
| Potential Energy (J) | | | |
| Difference KE - PE | | | |

Question 3-2: How well does the kinetic energy agree with the potential energy? Discuss any differences in terms of your experimental errors.

PLEASE CLEAN UP YOUR LAB AREA!

TURN OFF THE PHOTOGATE TIMERS.