Final Examination

Time limit: 3 hours. Do all 6 problems.

The exam is "closed book". You are allowed to bring one 8.5 inch × 11 inch sheet of paper with whatever you want written on one side into the exam with you.

1. Let U be a unitary operator on a Hilbert space $\mathtt{H}$:

   a) Show U is bounded with $||U|| = 1$.

   Suppose $\left.\begin{matrix} Uf = \lambda f \\ Ug = \mu g \end{matrix}\right\}$, $\lambda \neq \mu$, where f, g are vectors in $\mathtt{H}$ and λ and μ are complex numbers:

   b) Show that $|\lambda|^2 = |\mu|^2 = 1$;

   c) Show that $\left(f, g\right) = 0$.

2. Apply the Schmidt orthogonalization method to the monomials 1 , x , x^2 , … in order to derive the first three (non-trivial) Laguerre polynomials $L_n(x)$, $n = 1, 2, 3$ defined by

   $$\int_0^\infty dx\, L_m(x)\, L_n(x)\, e^{-x} = \delta_{mn} , \quad n = 0, 1, 2, \ldots$$

   with $L_0(x) = 1$.

3. By analytically continuing the following functions (when appropriate), discuss the domain of analyticity, and the location and nature of any singularities of each:

   a) $f_1(z) = \int_0^\infty dt \exp(-zt)$;

   b) $f_2(z) = \int_0^\infty dt \frac{z}{z + t^2}$;

   c) $f_3(z) = \sum_{k=0}^\infty \frac{(-1)^k}{z^{2k}}$.

4. Evaluate the integral $\int_0^\infty \frac{dx}{1 + x + x^2}$ by the calculus of residues. (Hint: consider the integral $\oint_\Gamma dz \frac{\log z}{1 + z + z^2}$ on an appropriate contour Γ .)

5. Consider the operator defined by the integral kernel $K(x, y)$ on the space $L^2(0, \infty)$. If its eigenvalues are defined by

$$K\,\varphi_n = \eta_n\,\varphi_n$$

and if the operator is self-adjoint and compact,

a) Express the operator in spectral form.

b) Give expressions for Tr K and Tr K^2 in terms of the eigenvalues.

c) For the kernel $K(x, y) = \dfrac{(xy)^{1/2}}{x^4 + y^4 + 1}$ evaluate the sums

$$S_1 = \sum_{n=0}^{\infty} \eta_n \text{ and } S_2 = \sum_{n=0}^{\infty} \eta_n^2$$

in closed form. (**Hint**: consider using cylindrical coordinates if necessary.)

6. A positive operator $A \geq 0$ is a Hermitian operator for which $\left(\Phi, A\Phi\right) \geq 0$ for any vector Φ in the Hilbert space. We say an operator $A \geq C$ if both are Hermitian and

$$\left(\Phi, \left(A - C\right)\Phi\right) \geq 0\,.$$

Using the trial vector

$$\Phi = \Psi - \sum_{n=1}^{N} c_n\,\varphi_n\,,$$

where the $\left\{ \varphi_n \right\}_{n=1}^{N}$ is any set of N vectors in $\mathcal{H}$, and Ψ is an arbitrary vector, show that

$$A \geq \sum_{m=1}^{N} \sum_{n=1}^{N} \left(A\varphi_m\right) B_{mn} \left(A\varphi_n\right)^{\dagger}\,,$$

where

$$\sum_{k=1}^{N} B_{mk} \left(\varphi_k\,, A\varphi_n\right) = \delta_{mn}\,.$$

(Hint: minimize with respect to the complex coefficients c_n.)